SOME STUDIES IN CERTAIN TYPES
OF
RINGS AND GROUP-RINGS

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CERTIFICATE

This is to certify that the thesis entitled " Some Studies in Certain Types of Rings and Group-Rings" by Shri J.B. Srivastava for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur is a record of bonafide research work carried out by him under my supervision and guidance for the last two years. The thesis has, in my opinion, reached the standard fulfilling the requirements of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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-:Note:-

Author is extremely sorry for making the thesis dirty and look odd but he could not help himself as he found a flaw in argument (when the thesis was complete in all respects) in one of his theorem (in section 2.2) because of which the whole of section 2.2 went wrong.

CONTENTS

CHAPTER - 0
INTRODUCTION

0.1: Notations and Symbols:

The bracketed numbers stand for the references in the bibliography at the end. Thus, for example, [3] refers to the third work in serial order listed in the bibliography which is arranged in alphabetical order of the last names of the authors.

Our cross-reference to results or definitions in the content of the thesis are numbered with three digits within a parenthesis, the first standing for the chapter concerned, the second for the section of that chapter in context, and the last one for the reference desired. The for example, (2.4.10) will mean the tenth result or definition in the fourth section of chapter two.

Section of each chapter are numbered with two digits, the first indicating the chapter concerned, and t he second its section in context. For example, 5.3 stands for the third section of the fifth chapter.

A number of the symbols used in this thesis are standard. Among these are: the symbols for set inclusion $A \supseteq B$, A includes B, $A \subseteq B$ A is contained in B; $a \in A$, a belongs to the set A; a/b, a divides b. Following is the list of some of the symbols and notations:

Z = ring of rational integers

$\mathbb{Q}$ = rational field

$H \leq G$ or $G \geq H$ = H is a subgroup of G.

$H \trianglelefteq G$ or $G \trianglerighteq H$ = H is a normal subgroup of G,

$J \trianglelefteq R$ or $R \trianglerighteq J$ = J is a two-sided ideal of R

$\cong$ = Isomorphic to

Char F = characteristic of the field F

R^R = left regular R-module

$M \dot{+} N$: external direct sum

$M \oplus N$: internal direct sum

A.C.C: ascending chain condition

D.C.C: descending chain condition

$\mathrm{Hom}_R(M,N)$ = additive group of R-homomorphisms of M into N.

$M \otimes_R N$ = tensor product of right R-module M and left R-module N.

M^G = induced module.

0.2. A Brief Survey of Problems:

The first three chapters of the thesis are devoted to the discussion of general ring theory while the last two chapters are devoted to the study of group-rings.

In Chapter I we give a characterization of 'direct-sum of division rings'. We have defined an associative ring R with unity 1 as DD-ring if it satisfies condition D given below:

Condition D:

If $r \neq 0$ in R, then there exist non-zero elements r^* and e^r in R such that

(i) $rr^* = r^*r = e^r = (e^r)^2$

(ii) $r.e^r = e^r.r = r$

and (iii) $r^*e^r = e^r.r^* = r^*$ "

We then have that every division ring and every external direct-sum of a finite number of division rings, are DD-rings. The problem here is to consider the converse of this i.e. when is a DD-ring, direct-sum of finite number of division rings. A DD-ring R is said to have the intersection property if every descending sequence of non-trivial ideals of the form Re^a has a non-trivial intersection. Then we have solved the problem to the extent stated in the following theorem:

"Every DD-ring R with intersection property is a unique (identically) direct-sum of division rings".

The importance of the theorem is because of importance of division rings in the study of Algebra.

In Chapter II, we present a general theory of Idealizers for groups and for rings, classify ideals in principal ideal domains in terms of their class numbers and give a characterization of irreducible k-manifolds. We define the Left-Idealizer of an element x in a set A (on which an algebraic composition of multiplication is defined) with respect to a subset S of A follows:

$$L_S(x) = \{y \text{ in } A: yx \text{ is in } S\}$$

Similarly we define the Right-Idealizer $R_S(x)$ and two-sided Idealizer $I_S(x)$.

In Section 2.2 we give a general theory of Idealizers for groups, introduce a topology Ω with the help of certain types of Idealizers such that $\{G, \Omega\}$ is a topological group. ~~We prove further that if G is nilpotent and Ω-connected then it must be abelian. Further we give some results about the relation between the connected component of identity and conjugacy classes of G.~~ In a similar way in Section 2.3 we give an Idealizer theory for rings and discuss its importance in the study of rings.

We define the class-number of an ideal S of a unitary ring R to be the number of distinct equivalence classes of R under the equivalence relation given by

For r_1, r_2 in R we say $r_1 \sim r_2$ iff $I_S(r_1) = I_S(r_2)$.

In a principal ideal domain, we have been able to classify the set of all ideals interms of their class numbers as follows:

In a principal ideal domain every ideal S can be written in the form $S = P_1^{m_1} \cdot P_2^{m_2} \cdot P_3^{m_3}$ ——— $P_t^{m_t}$ where P_i are distinct prime ideals then we have shown that the class number of S is $(m_1 + 1)(m_2 + 1)$ ——— $(m_t + 1)$. Two ideals have the same class-number iff they have the same number t of prime factors with the same set of indices $(m_1, m_2,$ ———, $m_t)$ occurring in some order

Finally in Section 2.5 we define a k-manifold with respect to a prime ideal k and prove "A k-manifold M is irreducible iff the ideal S* belonging to M is prime.

In Chapter III we discuss relative-projectivity and Property ϱ for general r ings. Let R be any unitary ring and P be a unitary subring of R such that R is a free right module over P with a basis $\{x_i : i \text{ in } I\}$ for some index-set I. We say $\{R, P\}$ has Property ϱ with respect to the basis x_i if $\sum x_i p_i \in$ Rad R implies each $p_i \in$ Rad P. In case the cardinality of I is finite we have shown that Property ϱ with respect to one basis implies the same with respect to any other basis. We have proved the transitivity of the Property ϱ . We have shown that Property ϱ is strongly related to relative-projectivity, which will be further supported when we discuss applications to group rings in Chapter V.

In Chapter IV we deal in details with different types augmentation maps and augmentation ideals to study groups and group rings, which show the power of augmentation theory in the study of groups and group rings.

If $A = RG$ is the group ring of G over R and $A = A_1 \trianglerighteq A_2 \trianglerighteq, A_3 \trianglerighteq$ ----, the lower central series of A then we have proved nth inverse image under augmentation of this series contains the nth term of the lower central series of G. We obtain several important results about central chains of groups and group rings. Then we characterize the inverse image under augmentation map of any ideal contained in the Magnus ideal of an integral group ring having only trivial units in terms quasi-regular elements.

In Section 4.4 we generalize the notions of different types of augmentation maps with respect to a subgroup of the group of all automorphisms of the group G containing the normal subgroup of all inner automorphisms of G. We have proved duality theorem which enables us to consider only upper and lower augmentations instead of left (right) upper and left (right) lower augmentation. In Section 4.5 we give the homology and cohomology theories of special types of augmented group rings.

In Chapter V we have shown that Property ρ and projective pairing are strongly related to each other. We have proved a sort of converge to Clifford's theorem see Curtis and Riener [6] . In the last section we prove a theorem on character kernels of discrete groups as a corollary to which follows the result of D.S. Passman [16] .

CHAPTER - I
ON DIRECT-SUMS OF DIVISION RINGS

1.1. INTRODUCTION:

The main purpose of this chapter is to give a characterization of 'direct-sum of division rings'. Through out this chapter R will stand for an associative ring with unity 1. We say that R is a DD-ring if it satisfies condition D given below.

Condition D:

"If $r \neq 0$ in R, then there exist non-zero elements r^* and e^r in R such that

(i) $rr^* = r^*r = e^r = (e^r)^2$

(ii) $r.e^r = e^r.r = r$

(iii) $r^*.e^r = e^r.r^* = r^*$.

We then have that every division ring and every external direct-sum of a finite number of division rings, are DD-rings. The problem here is to consider the converse of this i.e. when is a DD-ring a direct sum of finite number of division rings. First we prove some elementary properties about DD-rings. We show that a DD-ring has no non-trivial nilpotent elements. Then we easily get that every homomorphic image and every left (right) ideal of a DD-ring is again a DD-ring. Further we get that every minimal left ideal L in DD-ring R is of the form $L = R.e^a$ for every non-zero element a in L. Applying these results we have an important theorem (1.2.13 in next section)

"For any $a \neq 0$ in a DD-ring R, the left ideal $L = Re^a$ is minimal iff e^aRe^a is a division ring".

Proceeding further we obtain that every one-sided minimal ideal in a DD-ring is two-sided and two minimal ideals in a DD-ring R are R-isomorphic iff they are identical.

A DD-ring R is said to have the intersection property if every descending sequence of non-trivial ideals of the form Re^a has a non-trivial intersection. Then we prove that a DD-ring with intersection property contains a minimal ideal. We have then that every minimal ideal is an R-direct summand of R. Finally we prove the main result of this chapter as follows:

"Every DD-ring R with intersection property is a unique (identically) direct-sum of division rings".

Section 1.2: On Direct-sums of Division Rings:

Let R be a ring with its Jacobson radical $J(R)$ which is the intersection of the maximal left ideals, equivalently $J(R)$ consists of all x in R such that for all y in R, 1-xy has a left inverse. In fact $J(R)$ is also the intersection of the maximal right ideals and $J(R)$ consists of all x in R such that for all y in R, 1-xy has a right inverse i.e. $J(R) = \{x \text{ in } R: 1-xy \text{ is a unit for all } y \text{ in } R\}$.

Definition: R is called semi-primitive (semi-simple in the sense of Jacobson) if $J(R) = 0$.

The following standard example of a semi-primitive ring is often quoted:

1.1.1: "Let R be the external direct sum of n division rings Δ_i and $R = \dot{+} \sum_{i=1}^{n} \Delta_i$, $n < \infty$. Then R is a semi-primitive ring".

In fact R has minimum-condition for its ideals, and more particularly, R has no non-trivial nilpotent elements. Further let r be an element of R. Then $r = (r_1, r_2, \ldots\ldots, r_n)$ where $r_i \in \Delta_i$ for $i = 1, 2, \ldots\ldots, n$ respectively. Define $r^* = (r_1^*, r_2^*, \ldots\ldots, r_n^*)$ by the condition that

1.2.2:
$$r_i^* = \begin{cases} 0 \text{ if } r_i = 0 \\ r_i^{-1} \text{ if } r_i \neq 0 \end{cases}$$

Also define $e^r = (e_1^r, e_2^r, \ldots\ldots, e_n^r)$ by the condition that

1.2.3:
$$e_i^r = \begin{cases} 0 \text{ if } r = 0 \\ 1_i \text{ if } r_i \neq 0 \text{ where } 1_i \text{ is the identity element of } \Delta_i . \end{cases}$$

Then it is easily computed that any element r in R satisfied the folling conditions which for convenience, we shall refer to as condition (D):

1.2.4: "If $r \neq 0$ in R, then there exist non-zero elements r^* and e^r in R such that

(i) $rr^* = r^*r = e^r = (e^r)^2$,

(ii) $r.e^r = e^r.r = r$

(iii) $r^*.e^r = e^r.r^* = r^*$. "

We atonce have

Leema 1.2.5: r^* and e^r are uniquely determined for any $r \neq 0$ in R.

Proof: Let r_1^* and e_1^r be any two corresponding elements satisfying (i), (ii) and (iii) above, then

$$e_1^r.e^r = r_1^* re^r = r_1^* (re^r) = r_1^* r = e_1^r$$

$$= e_1^r.r\ r^* = (e_1^r r).r^* = rr^* = e^r$$

so $e_1^r = e^r$.

Also then $r_1^* = r_1^* e^r = r_1^*\ rr^* = (r_1^* r)r^* = e^r r^* = r^*$

Q.E.D.

Definition 1.2.6: A ring R will be called a DD-ring if it satisfies condition (D).

From the example above, it is clear that every division ring and every external direct-sum of a finite number of division rings, are DD-rings. We prove below some elementary properties of DD-rings.

Theorem 1.2.7: A DD-ring R has no non-trivial nilpotent elements.

Proof: Suppose r in R has $r^n=0$ for some finite positive integer n, then $r^n.r^* = r^{n-1}(rr^*) = r^{n-1}.e^r = r^{n-1} = 0$

In a finite number of steps we show that $r = 0$ in R.

Corollary 1.2.8: A DD-ring has no non-trivial nilpotent one-sided or two-sided ideals.

Theorem 1.2.9: Every homomorphic image and every left (right)-ideal of a DD-ring is a DD-ring.

Proof: Let f be an onto homomorphism of R onto another ring S.

Then given s in S, there exists r in R such that $f(r) = s$.
Put $s^* = f(r^*)$ and $e^s = f(e^r)$

It is easily verified that s, s^*, e^s satisfy condition (D).
Thus S is a DD-ring.

Next let L be a left ideal of R. If $L = (0)$, then there is nothing to prove. If $L \neq (0)$, then let k in L and $k \neq 0$. There exist k^* and e^k in R satisfying condition (D). Thus $k^*k = e^k$ is in L and also $k^* = k^*.e^k$ is in L. Thus every element in L satisfies condition (D) in L itself, so L is a DD-ring in its own right.
Similarly we can prove that every right ideal in R is a DD-ring.

Q.E.D.

Definition 1.2.10: A subring T of a ring R is said to be "semi-central" if for all elements x,y in T, there exists an element $C(x,y)$ in T such that $xy = C(x,y)\,x$.

Then we prove

Theorem 1.2.11: Every minimal left-ideal L in a DD-ring R is semi-central and has the form $L = R.e^{a}$ for every non-zero element a in L.

Proof: If x,y are non-zero elements of L, then by Theorem 2, x^*, e^x, y^*, e^y are all in L. Hence $R.e^x \subseteq L$. By the minimality of L, either $Re^x = 0$ or $Re^x = L$.

Since $e^x.e^x = e^x \neq 0$ is in Re^x, so $Re^x = L$.

Now, since e^x is an idempotent element, so it is a right unity in L. Hence, $xy = xye^x = xyx^*x = C(x,y)x$ where $C(x,y) = xyx^*$, is in L again.

Thus L is semi-central.

Q.E.D.

Corollary 1.2.12: The endomorphic images of the additive group $(R,+)$ under right multiplication by the elements of a minimal left ideal L, are invariant under these endomorphisms.

Proof: If a,b are any two elements in L, then $C(a,b)$ is defined in L such that $ab = C(a,b).a$

Hence $(Ra)\,b = \left[\, R.\,C(a,b) \,\right] a \subseteq R.a$

Thus each Ra for a in L is in variant under right-multiplication by any element b in L.

Q.E.D.

From above, we have the following theorem:

Theorem 1.2.13: For any $a \neq 0$ in a DD-ring R, the left ideal $L = Re^{a}$ is minimal iff $e^{a}Re^{a}$ is a division ring.

Proof: First let $e^a he^a$ be a division ring. Let M be a left ideal in L and $m \neq 0$ is in M. We assert that $e^a.m \neq 0$. For, let $e^a m=0$. Then $e^a(m.e^q) = 0$, as e^a is a right unity in L. Hence $me^a.me^a = m\,(e^a me^a) = 0 \Rightarrow m^2 = (me^a)^2 = 0$.

But by theorem 1.2.7, R has no non-trivial nilpotent elements. Hence $m = 0$ contrary to our assumption that $m \neq 0$. Thus $e^a m \neq 0$ if $m \neq 0$.

Now let $x = e^a m = e^a me^a$ which is therefore a non-zero element of the division ring $e^a Re^a$. Hence it has an inverse x' in $e^a Re^a$ such that $x'x = e^a$, the unity in $e^a Re^a$.

Therefore e^a is in M so that $Re^a = L = M$. Thus L is a minimal left ideal in R.

Conversely let $L = Re^a$ be a minimal left ideal in R and $e^a xe^a$ be a non-zero element in $e^a Re^a$, since $e^a Re^a \subseteq L$ so $R(e^a xe^a) \subseteq RL \subseteq L$. But $R(e^a xe^a)$ is a left ideal in R and $e^a(e^a xe^a) = e^a xe^a \neq 0$ in it, so in view of the minimality of L we have $R(e^a xe^a) = L$. Hence $(Re^a)\,(e^a xe^a) = L$ so that there exists x' in R $(x'e^a)\,(e^a xe^a) = e^q$. Then $(e^a x'e^a)\,(e^a xe^a) = e^a.e^a = e^a$, so that $e^a x'e^a$ is the inverse of $e^a xe^a$. Thus $e^a Re^a$ is a division ring.

Q.E.D.

Similarly, we can prove that

Theorem 1.2.14: A right ideal $I = e^a R$ for some non-zero element of the DD-ring R, is minimal iff $e^a Re^a$ is a division ring.

We then have:

Theorem 12.15: Every one-sided minimal ideal of a DD-ring R is two-sided.

Proof: Let L be a minimal left ideal of R. By theorem 1.2.11, $L = Re^a$ for a non-zero element a in L. By theorem 1.2.13, e^aRe^a is a division ring. Then by theorem 1.2.14 $I = e^aR$ is a minimal right ideal in R.

By theorem 1.2.11, if $x \neq 0$ in L, then $L = Re^x$ also. Hence $e^x.e^a = e^x$ and $e^a.e^x = e^a$. But $e^a.e^a = e^a$. Hence $e^ae^x - e^ae^a = e^a - e^a = 0 = e^a(e^x - e^a)$.

Then $(e^x - e^a)\ (e^x - e^a) = e^x(e^x - e^a) - e^a\ (e^x - e^a)$

$$= e^x.e^x - e^x.e^a = e^x - e^x = 0$$

By theorem 1.2.9, $e^x - e^a = 0$ or $e^x = e^a$

Hence e^x is in I for every non-zero x in L.

Therefore $x = e^x x$ is in I so that $L = I$, so L is two-sided and I too.

Q.E.D.

From the proof we have

Corollary 1.2.16: If L is minimal ideal in a DD-ring R then $e^x = e$ is the same for each $x \neq 0$ in L.

Corollary 1.2.17: If $L = R.e^a$ is a minimal ideal in a DD-ring R, then $L = e^aRe^a$ is a division ring.

Proof: $Re^a = L = I = e^aR$, thus e^a is a two-sided unity in L, so that $L = e^aL = e^aRe^a$ is a division ring by theorem 1.2.13.

Q.E.D.

More generally we have

Theorem 1.2.18: Every left ideal L in a DD-ring R which has the form $L = R.e^a$ for some non-zero a in R, is a two-sided ideal of the form $L = e^aRe^a$

Proof: Let $M = \{x \text{ in } L : e^a x = 0\}$. Since $M \subseteq L$ and e^a is a right unity in L, so $Me^a = M$. Then $M^2 = (Me^a)(Me^a) = M(e^aM)e^a = 0$ but by corollary 1.2.8, it then follows that $M = 0$.

Now for any k in L, $e^a(k-e^ak) = 0$ which now implies that $k-e^ak = 0$ or $k = e^ak$. Thus e^a is a two sided unity in L and $L = Re^a = e^aL = e^aRe^a$.

Now if $I = e^aR$ is the corresponding right ideal, then as above we can show that $I = e^aRe^a$ also.

Hence $L = I$ and both are therefore two-sided ideals in R.

Q.E.D.

Theorem 1.2.19: Two minimal ideals in a DD-ring R are R-isomorphic iff they are identical.

Proof: Let $f: L_1 \to L_2$ be an R-isomorphism of a minimal ideal L_1 onto a minimal ideal L_2 in a DD-ring R. Suppose $L_1 = Re^a$ by theorem 1.2.11 for some non-zero a in L_1.

Let $f(e^a) = x$ in L_2 then $f(e^a) = x = f(e^ae^a) = e^a f(e^a) = e^a.x$

By theorem 1.2.15 L_1 and L_2 are two-sided so we have

$x = e^ax$ is in L_1

By theorem 1.2.9 then e^x is in L_1 and by theorem 1.2.11 $L_2 = R.e^x \subseteq L_1$

Similarly $L_2 \supseteq L_1$ and equality follows

Q.E.D.

Corollary 1.2.20: If L is a minimal ideal in a DD-ring R, then for every $a \neq 0$ in L, $e^a Re^a$ gives the same division ring.

Definition 1.2.21: A DD-ring R is said to have the intersection-property if every descending sequence of non-trivial ideals of the form Re^a has a non-trivial intersection.

We then start proving:

Theorem 1.2.22: If R is a DD-ring with intersection property, then R contains a minimal ideal.

Proof: Let $E = \{ e^a : a \neq 0 \text{ in } R \}$ be the set of all idempotents of the form e^a defined by condition (D) for each a in R.

We introduce an equivalence and a partial order relations in E as follows:

We shall say that $e^a \equiv e^b$ if $Re^a = Re^b$, and $e^a \leq e^b$ if $Re^b \leq Re^a$.

We verify that

(i) $e^a \leq e^b$ iff $e^b e^a = e^b$ since e^a is a right unity in $Re^a \supseteq Re^b$

(ii) $e^a \leq e^a$ (Reflexivity)

(iii) $e^a \leq e^b$ and $e^b \leq e^a$ implies that $Re^a = Re^b$ implies $e^a \equiv e^b$.

This proves anti-symmetry

(iv) $e^a \leq e^b$ and $e^b \leq e^c$

$Re^c \leq Re^b \leq Re^a$ implies $Re^c \leq Re^a$ implies $e^a \leq e^c$.

This proves transitivity.

Now if $e^{a_1} \leq e^{a_2} \leq \ldots\ldots\ldots$ is a chain in E then

$Re^{a_1} \supseteq Re^{a_2} \supseteq \ldots\ldots$ is a descending sequence of non-empty left ideals in R. Therefore $L = \cap Re^{a_i} \neq 0$.

So by theorem 1.2.9, $k \neq 0$ in L implies e^k is in L so that $Re^k \subseteq L \leq Re^{a_i}$ for each i. Thus e^k is an upper bound for this chain. So by Zorn's lemma, we obtain a maximal element e^a in E.

We assert that $M = Re^a$ is a minimal ideal in R. For otherwise let N be a non-trivial left ideal in M. If $n \neq 0$ in N, then by theorem 1.2.9, e^n is in N and $Re^n \leq N \leq M = Re^a$ this will imply $e^a \leq e^n$ contradicting the maximality of e^a in E. So M is a minimal ideal in R.

Q.E.D.

Theorem 1.2.23: Every minimal ideal of a DD-ring R is an R-direct summand of R.

Proof: Let $L = Re^a$ be a minimal ideal in R. Note that by theorem 1.2.15, L is two-sided. Define $f : R \longrightarrow L$ by $f(r) = re^a$. f is an R-homomorphism of R onto L since L is minimal. For any r in R, we may write $r = r.e^a + (r - re^a)$ where re^a is in L and $f(r-re^a) = f(r) - f(re^a)$

Note also that for any x in L $f(x) = xe^a = x$, since e^a is a right unity in L. Thus $f(r-re^a) = re^a - re^a = 0$. So $r-re^a$ belongs to the Kern. f.

Therefore $R = L + \text{Kern. } f$

Since Kern. f is also a left ideal in R, so L Kern. f is a left ideal contained in L, since $f(e^a) = e^a \neq 0$, so $L \cap \text{Kern } f \neq L$ hence $L \cap \text{Kern } f = (0)$ by the minimality of L and hence $R = L \oplus \text{Kern } f$.

Q.E.D.

Now we shall prove our main structure theorem of this section.

Theorem 1.2.24: Every DD-ring R with intersection property is a unique (identically i.e. not only upto isomorphism) direct sum of division rings. Conversely, every direct sum of division rings is a DD-ring.

Proof: The converse part follows from the construction in the beginning of this section.

To prove the other part let $\{L_i\}$ be the collection of all minimal ideals of the given DD-ring R. Then $\oplus \sum L_i$ is an ideal in R and by an application of theorem 1.2.23 it is a direct summand of R. Now if $R = \oplus \sum L_i \oplus S$, and S is non-trivial, then by theorem 1.2.9 S is itself a DD-ring, since every collection of ideals in S is a collection of ideals in R, hence S itself has the intersection property. Then by theorem 1.2.22, S contains a minimal ideal L^*. By theorem 1.2.23, L^* is a direct summand of S and hence of R. But then L^* belongs to the set $\{L_i\}$ which leads to a contradiction. Therefore $R = \oplus \sum L_i$.

To treat the uniqueness part let $R = \oplus \sum L_i = \oplus \sum L_j'$. Define π_i to be the projection of L_i'. Then π_i restricted to L_j cannot be trivial for each j. Suppose π_i restricted to L_i is non-trivial. Then for the minimality of L_i and L_i' we conclude that $L_i \cong L_i'$. Then by theorem 1.2.19, it follows that $L_i = L_i'$. Thus if $S = \{L_i\}$ and $T = \{L_i'\}$ then $S \subseteq T$ and similarly $T \subseteq S$ and hence $T = S$ which proves the uniqueness of the direct summands of R.

Q.E.D.

CHAPTER - II
THEORY OF IDEALIZERS

2.1. INTRODUCTION:

The main purpose of this chapter is to give a general theory of Idealisers for an arbitrary set A on which an algebraic composition of multiplication is defined, in particular for groups and rings. For a set A on which an algebraic composition of multiplication is defined, we define the Left-Idealiser of x in A with respect to a subset S of A as

$$L_S(x) = \{y \text{ in } A : y.x \text{ is in } S\} .$$

Similarly we define the Right-Idealizer $R_S(x)$ and two-sided Idealizer $I_S(x)$.

In Section 2.2 we have developed a theory of Idealizers for a given group G. With the help of certain type of Idealizers, we have introduced a topology Ω in G such that $\{G, \Omega\}$ becomes a topological group. Moreover the topology enjoys a special property that arbitrary intersections of open sets in $\{G, \Omega\}$ are open. We have called topological spaces with this property I-spaces and given a characterization of Hausdorff I-spaces. We have denoted $\{G, \Omega\}$ as I-group. We have been proved that a non-abelian connected I-group has a trivial centre. It follows, then the main result of this section, that a connected nilpotent I-group is always abelian. Then some other important results relating the connected component of identity and conjugacy classes are also obtained.

In Section 2.3 we develope a parallel theory for rings. We introduce a topology Ω in a ring A with the help of Idealizers. Besides other results, we get a good result Theorem 2.3.15 "If x and y are unit in A and $x \neq y$ then these elements can be separated in A T_1-manner iff one is not an integral multiple of the other". In this case also $\{A, \Omega\}$ is an I-space. So many other good results have been obtained.

Section 2.4 deals with the class number of an ideal defined with the help of Idealisers. Some important results obtained are as follows:

(i) The class number of an ideal in 2 iff it is prime.

(ii) If $S = P^n$, where P is a prime ideal of a principal ideal domain R, then the class number of S is $n+1$.

(iii) If $S = P_1.P_2.....P_t$, where P_i are distinct prime ideals in a principal ideal domain R, then the class number of S is equal to 2^t.

(iv) As above $S = P_1^{m_1} P_2^{m_2} .....P_t^{m_t}$ implies class number of S is $(m_1+1)(m_2+1).....(m_t+1)$.

(v) Two ideals in a principal ideal domain have the same class number iff they have the same number t of prime factors with the same set of indices $(m_1, m_2, ..., m_t)$ occurring in some order.

Thus in a principal ideal domain, we have been able to classify the set of all ideals interms of their class number.

Finally in the last Section 2.5, we define a k-manifold and prove an important Theorem 2.5.2 that "A k-manifold M is irreducible iff the ideal S^* belonging to M is prime".

2.2. Theory of Idealisers for Groups:

Let A be a set on which an algebraic composition of multiplication is defined. Let S be a subset of A. For any x in A, we define the Left-Idealiser of x with respect to S as

2.2.1. $L_S(x) = \{y: yx \text{ is in } S\}$

Similarly we define the Right-Idealiser $R_S(x)$. The two-sided Idealiser $I_S(x)$ is defined by

2.2.2. $I_S(x) = \{y \text{ in } A \text{ such that } y.x \text{ and } xy \text{ are both in } S\}$.

Clearly we have:

2.2.3. $I_S(x) = L_S(x) \cap R_S(x)$.

In case the multiplication is commutative, we get

2.2.4. $I_S(x) = L_S(x) = R_S(x)$.

Proposition 2.2.5: If A is a group then $I_S(x) = L_S(x) = R_S(x) = S$ for every x in S iff S is a subgroup of A.

Proof: Suppose $I_S(x) = L_S(x) = R_S(x) = S$ for every x in S.

Suppose y is in S then $y \in R_S(x)$ so xy belongs to S i.e. for every pair of elements x,y in S we have xy is in S. Further we have 1 is in $I_S(x)$ implies 1 is in S.

Now $x^{-1}x = xx^{-1} = 1$ is in S so x^{-1} is in $I_S(x) = S$. Hence S is a subgroup of A.

Conversely if S is a subgroup of A

$L_S(x) = \{y \text{ in } S: yx \text{ is in } S\}$ but S is a subgroup so if x is in S then yx in S implies y is in S.

So $L_S(x) \subseteq S$

but we have $S \subseteq L_S(x)$ because y in S implies yx in S whenever x in s implies y in $L_S(x)$.

Thus $L_S(x) = S$. Similarly others.

Q.E.D.

Next let $\{S_i\}$ be the class of all subsets of the set A containing x where i is in some indexing set. Then we have the duality-relations:

Lemma 2.2.6: (i) $\cap L_{S_i}(x) = L_{\cap S_i}(x)$

and (ii) $\cup L_{S_i}(x) = L_{\cup S_i}(x)$

Similarly for $R_{S_i}(x)$ and $I_{S_i}(x)$.

Now we shall develop a theory of Idealisers for a given group G. In order to construct a topology for G such that G becomes a topological group, we go to generalize the notion of normaliser of a subgroup. For a semi-group $S \neq (1)$ of a group G and a subset T of G define $L_S(T) = \{x \text{ in } G: xT \subseteq S\}$ and similarly $R_S(T)$ and $I_S(T)$.

Lemma 2.2.7: If y is in $\cap L_{S_i}(T_i)$, then there exists S and T, such that y is in $L_S(T)$ which is contained in $\cap L_{S_i}(T_i)$.

Proof: Consider $S = \{1, \cap S_i\}$. This is a semi-group with unity.

Let $T = \{y^{-1}\}$. Consider $L_S(y^{-1})$.

Since $yy^{-1} = 1$ is in S, this implies y is in $L_S(y^{-1})$.

Also z in $L_S(y^{-1})$ implies $zy^{-1} = 1$ or $zy^{-1} = s$ for some s in $\cap S_i$

i.e. Either z=y or z=sy

$zT_i = yT_i$ or $zT_i = syT_i = syT_i$

and so in both the cases $zT_i \subseteq S_i$. Since this happens for every i so z is in $L_{S_i}(T_i)$. We have our result

y belongs to $L_S(T) \subseteq \cap L_{S_i}(T_i)$.

Q.E.D.

$\cancel{x \quad xL_{S_1}(T_1) = x \quad L_{S_1}(T_1) \quad V_1}.$

~~So by Lemma 2.2.7 there exist S and T with property~~

$\cancel{x \quad xL_S(T) \quad x \quad L_{S_1}(T_1) \quad V_1}.$

~~This implies V_1 is open and hence is in~~ .

Q.E.D.

More generally, let any topological space with the property that arbitrary intersections of open sets is open, be called an Idealiser-space or in short an I-space. Then we have a strong conclusion.

Lemma 2.2.10: An I-space is Hausdorff iff every pair of disjoint sets can be separated by disjoint open sets.

Proof: Let P_1 and P_2 be disjoint subsets of the I-space (X, Ω) which is Hausdorff. Choose an x in P_1, then for every y in P_2 we have $x \neq y$ (because $P_1 \cap P_2 = \varphi$). By the Hausdorff separation axiom there will exist $V_x^{(y)}$ and V_y in Ω such that x is in $V_x^{(y)}$ and y will be in V_y, further $V_x^{(y)} \cap V_y = \varphi$. Therefore we have

x is in $\bigcap_{y \in P_2} V_x^{(y)} = V_x$ belonging to Ω (V_x is in Ω because (X, Ω) is an I-space) and $P_2 \subseteq \bigcup_{y \in P_2} V_y = V_x^{(P_2)}$ belonging to Ω such that $V_x \cap V_x^{(P_2)} = \varphi$.

Now for each x in P_1, choose V_x and corresponding $V_x^{(P_2)}$ as above.

Then $P_1 \subseteq \bigcup_{x \in P_1} V_x = V_1$ in Ω.

and $P_2 \subseteq \bigcap_{x \in P_1} V_x^{(P_2)} = V_2$ in Ω.

and $V_1 \cap V_2 = \varphi$.

Since 1 is in S so 1 is in V.

Also z_1, z_2 belong to V implies $z_1 = xs_1^{-1}$ and $z_2 = xs_2x^{-1}$ where s_1, s_2 are in S.

$$z_1z_2 = xs_1x^{-1}xs_2x^{-1} = xs_1s_2x^{-1} = xsx^{-1} \text{ where } s = s_1s_2 \text{ belongs to S.}$$

This implies z_1z_2 belongs to V. So V is a semi-group with 1 and so $V = L_V(1)$ is in M.

And we have $x^{-1}Vx \quad S \quad L_S(T) = U$ in M.

Thus we have proved that for every U in M and x in G there exists a V in M such that $x^{-1}Vx \quad U$.

(d) Finally, let $U = L_S(T)$ be in M, x in U

Define $V = \; y$ in S; yx is in S

y_1, y_2 in V implies y_1x and y_2x are in S implies $y_1y_2x = y_1(y_2x) = y_1s_1$ where y_1, s_1 are in S implies y_1y_2x is in S' implies y_1y_2 belongs to V .

Thus $V = \quad 1, V$ is a semi-group with 1 and we have

$L_V(1) = V$ is in M and $V_x \quad S \quad U$.

Thus for all U in M, x in U there exists a V in M such that $V_x \quad U$.

a , b , c and d imply that G, is a topological group.

Q.E.D.

Proposition 2.2.9: Arbitrary intersections of open sets in G, are open.

Proof: Let V_i , V_i be such an intersection and let x belongs to V_i.

Now x in V_i implies there exists an $L_{S_i}(T_i)$ a neighbourhood of identity such that x belongs to $xL_{S_i}(T_i) \quad V_i$ (because V_i is open). This will be true for each i.

~~$x \quad xL_{S_1}(T_1) = x \quad L_{S_1}(T_1) \quad V_1$.~~

~~So by Lemma 2.2.7 there exist S and T with property~~

~~$x \quad xL_S(T) \quad x \quad L_{S_1}(T_1) \quad V_1$~~

~~This implies V_1 is open and hence is in~~ .

Q.E.D.

More generally, let any topological space with the property that arbitrary intersections of open sets is open, be called an Idealiser-space or in short an I-space. Then we have a strong conclusion.

Lemma 2.2.10: An I-space is Hausdorff iff every pair of disjoint sets can be separated by disjoint open sets.

Proof: Let P_1 and P_2 be disjoint subsets of the I-space (X,Ω) which is Hausdorff. Choose an x in P_1, then for every y in P_2 we have $x \neq y$ (because $P_1 \cap P_2 = \varphi$). By the Hausdorff separation axiom there will exist $V_x^{(y)}$ and V_y in Ω such that x is in $V_x^{(y)}$ and y will be in V_y, further $V_x^{(y)} \cap V_y = \varphi$. Therefore we have

x is in $\bigcap_{y \in P_2} V_x^{(y)} = V_x$ belonging to Ω (V_x is in Ω because (X,Ω) is an I-space) and $P_2 \subseteq \bigcup_{y \in P_2} V_y = V_x^{(P_2)}$ belonging to Ω such that $V_x \cap V_x^{(P_2)} = \varphi$.

Now for each x in P_1, choose V_x and corresponding $V_x^{(P_2)}$ as above.

Then $P_1 \subseteq \bigcup_{x \in P_1} V_x = V_1$ in Ω.

and $P_2 \subseteq \bigcap_{x \in P_1} V_x^{(P_2)} = V_2$ in Ω.

and $V_1 \cap V_2 = \varphi$.

2.3. Theory of Idealisers for Rings:

In this section we shall develop a general theory of Idealisers for unitary rings. From the definitions of the previous section we easily have

Theorem 2.3.1: If S is an additive subgroup of a unitary ring A then $I_S(x)$ is also so. Also if S is an additive subgroup of A then S is a $\dfrac{\dfrac{\text{left}}{\text{right}}}{\text{two-sided}}$ ideal of A iff $\dfrac{\dfrac{L_S(x)}{R_S(x)}}{I_S(x)} = A$. Also $L_s(0) = A$ and $L_S(1) = S$.

Now let A be a unitary ring. Given a subset S in A define $V_S(x) = L_S(x) \cup \{x\}$ for every x in S. Let Ω be the class of null set and the arbitrary unions of the sets $V_S(x)$ for all possible S and x. We prove

Proposition 2.3.2: If y is in $\cap V_{S_i}(x_i)$, then there exists $V_S(z)$ such that y belongs to $V_S(z)$ contained in $\cap_i V_{S_i}(x_i)$.

Proof: Let $S = \{y, 1\}$ then y is in $V_S(1) \subseteq \cap V_{S_i}(x_i)$

Q.E.D.

Proposition 2.3.3: Arbitrary intersections and unions of elements of Ω are in Ω.

Proof: About the unions the proposition is obvious.

Let $\cap V_i$, V_i in Ω be an arbitrary intersection

Then x in $\cap V_i$ implies there exists $V_{S_x}(z_x)$ such that x is in $V_{S_x}(z_x) \subseteq \cap V_i$.

Thus $\cup V_{S_x}(z_x) \subseteq \cap V_i$ but $\cap V_i \subseteq \cup V_{S_x}(z_x)$ so $\cap V_i = \cup V_{S_x}(z_x)$.

Q.E.D.

Corollary 2.3.4: $\{A, \Omega\}$ is a topological space with topology Ω for which the $V_S(x)$ form a base.

Corollary 2.3.5: V in Ω iff for every x in V, there exists $V_S(z)$ such that x belongs to $V_S(z) \subseteq V$.

Proof: V in Ω implies $V = \cup V_S(z)$

$\therefore$ x in V implies x in $V_S(z)$ for some S and z.

Conversely let x be in V implies that x is in $V_{S_x}(z_x) \subseteq V$

$$\therefore V \subseteq \bigcup_{x \in V} V_{S_x}(z_x) \subseteq V$$

So $V = \bigcup_{x \text{ in } V} V_{S_x}(z_x)$ i.e. V belongs to Ω .

Q.E.D.

Now since 1 is in every $V_S(x)$ for all choice of S and x.

So 1 belongs to V for every V in Ω .

Corollary 2.3.6: $\{A, \Omega\}$ is not T_0.

Proof: Since 1 can not be separated by elements of Ω .

But we can show that

Proposition 2.3.7: $A - \{1\}$ with the induced topology is a discrete space.

Proof: z is in $A - \{1\}$ the $z = V_{\{z,1\}}(1) \cap [A - \{1\}]$ is open in $A - \{1\}$

Q.E.D.

This shows that in order to get any new information about the algebraic structure of A we have to put sufficient extra conditions on the sets S in $V_S(x)$. In any case we have

Theorem 2.3.8: ~~If A is also a vector space over a field then A, is a linear topological space.~~

Proof: ~~Follows in the similar lines as that of theorem 2.2.8 in the previous section.~~

Now let $\{S_i\}$ be the class of the additive subgroups of A.

For any subset T of A define

2.3.9 $\quad L_{S_i}(T) = \{y \text{ in } A: yT \subseteq S_i\}$

Proposition 2.3.10: Every $L_S(T)$ is an additive subgroup of A.

Proof: x,y in $L_S(T)$ implies $xT \subseteq S$ and $yT \subseteq S$ implies $(x-y)T \subseteq S$.

Since S ia an additive subgroup.

So $x-y \in L_S(T)$.

Q.E.D.

Proposition 2.3.11: y in $L_{S_1}(T_1) \cap L_{S_2}(T_2)$ then there exists an $L_S(T)$ with y in $L_S(T) \subseteq L_{S_1}(T_1) \cap L_{S_2}(T_2)$.

Proof: Take $S = L_{S_1}(T_1) \cap L_{S_2}(T_2)$ then S will be an additive subgroup of A containing y.

$\therefore$ y is in $L_S(1) = S = L_{S_1}(T_1) \cap L_{S_2}(T_2)$

Q.E.D.

Corollary 2.3.12: Arbitrary intersections of the $L_S(T)$ is again $L_S(T)$.

Corollary 2.3.13: The class of additive subgroups of A coincides with the $L_S(T)$'s.

Taking the set of all $L_S(T)$'s as the basis for a topology Ω, we obtain a topological space $\{A,\Omega\}$. Thus the study of $\{A,\Omega\}$ will enable us to investigate the structure of a ring with respect to its additive subgroups. Note that this is thus a generalization of the problem of investigating the intertwining of the multiplicative and additive groups of a division ring. Proceeding with our investigations we have.

Proposition 2.3.14: $V \in \Omega$ iff for every x in V there exists an $L_S(T)$ such that x is in $L_S(T)$ contained in V.

Note that 0 is in $L_S(T)$ for all S and T. $\{A, \Omega\}$ cannot be a T_1-space as such. But as our aim is to apply it to multiplicative groups in A, so we have the important result.

Theorem 2.3.15: If x and y are units in A and $x \neq y$ then these elements can be separated in a T_1-manner iff one is not an integral multiple of the other.

Proof: Only if part

We observe that x in V implies x in $L_S(T) \subseteq V$ for some S and T. $L_S(T)$ is additive subgroup so mx is in $L_S(T)$ for every integer m. Suppose now y = mx then whenever x is in V we have y is also in V. So they can not be separated in a T_1-manner.

If part: Suppose $y \neq x$ and x,y can not be separated in a T_1-manner.

Then for all $L_{S_x}(T_x)$ containing x, we must have y in $L_{S_x}(T_x)$ or for each $L_{S_y}(T_y)$ containing y, x is in $L_{S_y}(T_y)$. Assume that the first case occurs. Then y is in $L_{S_x}(T_x)$ for all S_x and T_x.
Take $S_x = \{0, \pm 1, \pm 2, \pm 3, \text{------}\}$ and $T_x = \{x^{-1}\}$. Then we see that x is in $L_{S_x}(T_x) = \{mx: m \text{ an integer}\}$

$\therefore$ y in $L_{S_x}(T_x)$ implies y = mx.

So x and y can be separated in a T_1-manner whenever $y \neq mx$.

Q.E.D.

If one takes U to be the group of units of A modulo integers i.e. if we identify all units that are integral multiples of the same unit, then we get the important corollary.

Corollary 2.3.16: With the topology Ω_U induced by Ω in A,

$\{U, \Omega_U\}$ is a T_1-space.

The most interesting topological property of $\{A, \Omega\}$ is given by

Theorem 2.3.17: Arbitrary intersections of elements of Ω are again in Ω i.e. $\{A, \Omega\}$ is an I-space.

Proof: x in $\bigcap V_i$ implies x is in V_i for every i. Thus for every i there exists S_i, T_i such that x is in $L_{S_i}(T_i) \subseteq V_i$.

So x is in $\bigcap L_{S_i}(T_i) \subseteq \bigcap V_i$. But $\bigcap L_{S_i}(T_i)$ is an additive subgroup so it is $L_S(T)$ for some S and T and we have

x in $L_S(T) \subseteq \bigcap V_i$.

Q.E.D.

Corollary 2.3.18: $\{U, \Omega_U\}$ is an I-space.

Corollary 2.3.19: $\{U, \Omega_U\}$ is Hausdorff iff it is normal.

Proof: $\{U, \Omega_U\}$ is T_1 so if it is normal then it is Hausdorff.

If $\{U, \Omega_U\}$ is Hausdorff then by lemma 2.2.10 of the previous section we have $\{U, \Omega_U\}$ is normal.

Q.E.D.

Now we shall give some simple observations about certain chain conditions:

Theorem 2.3.20: (i) If S is a left ideal of R, then $L_S(T)$ is a left ideal of R.

(ii) If S is a right ideal of R, then $R_S(T)$ is a right ideal of R.

(iii) If T is a left ideal of R, then $L_S(T)$ is a right ideal of R.

(iv) If T is a right ideal of R, then $R_S(T)$ is a left ideal of R.

Proof: (i) y in $L_S(T)$ implies $yT \subseteq S$ implies $(Ry)T = R(yT) \subseteq RS \subseteq S$.

(ii) y in $R_S(T)$ implies $Ty \subseteq S$ implies $TyR = Ty.R \subseteq SR \subseteq S$.

(iii) y in $L_S(T)$ implies $yT \subseteq S$ implies $yRT = y.RT \subseteq yT \subseteq S$.

(iv) $y$in R_S (T) implies $Ty \subseteq S$ implies $TRy = TR.y \subseteq Ty \subseteq S$.

Q.E.D.

Lemma 2.3.21: If $T_1 \subseteq T_2 \subseteq T_3 \subseteq$ ------ is an ascending chain of subsets of R and S is any subset of R, then we have $L_S(T_1) \supseteq L_S(T_2) \supseteq \cdots$

Similarly if $T_1 \supseteq T_2 \supseteq T_3 \supseteq$ ---------- then

$L_S(T_1) \subseteq L_S(T_2) \subseteq L_S(T_3) \subseteq$ ----------

Similarly for $R_S(T)$ and $I_S(T)$.

Lemma 2.3.22: If $S_1 \subseteq S_2 \subseteq S_3 \subseteq$ ------then

$$L_{S_1}(T) \subseteq L_{S_2}(T) \subseteq L_{S_3}(T) \subseteq \text{------}$$

Similarly if $S_1 \supseteq S_2 \supseteq S_3 \supseteq S_4 \supseteq$ -------- then

$$L_{S_1}(T) \supseteq L_{S_2}(T) \supseteq L_{S_3}(T) \supseteq \text{--------}$$

Similarly for $R_S(T)$ and $I_S(T)$.

These give rise to the following obseration:

"Let T be any subset of R and $S_1 \supseteq S_2 \supseteq S_3 \supseteq$ ------ be descending chain of left ideals of R. Then

$$L_{S_1}(T) \supseteq L_{S_2}(T) \supseteq L_{S_3}(T) \supseteq \text{-----}$$

is a descending chain of left ideals of R".

Theorem 2.3.23: (Isomorphism Theorem) Let $f: A_1 \longrightarrow A_2$, be an isomorphism of the ring A_1 onto the ring A_2. If T_1, $S_1 \subseteq A_1$ are left ideals and $S_2 = f(S_1)$, $T_2 = f(T_2)$, then f is an isomorphism of $L_{S_1}(T_1)$ onto $L_{S_2}(T_2)$.

Proof: Let

x in $L_{S_1}(T_1) = \{x \text{ in } A_1 : xT_1 \subseteq S_1\}$

so $f(x).f(T_1) = f(xT_1) \subseteq f(S_1) = S_2$

so $f(x).T_2 \subseteq S_2$ implies $f(x)$ in $L_{S_2}(T_2)$

Conversely let $z \in L_{S_2}(T_2) = \{y \text{ in } A_2 : yT_2 \subseteq S_2\}$

therefore $z.f(T_1) = z.T_2 \subseteq S_2$.

Since f is an isomorphism there exists x in A such that $f(x) = z$

therefore $f(x).f(T_1) = f(xT_1) \subseteq S_2 = f(S_1)$

therefore $f^{-1}f(xT_1) = xT_1 \subseteq f^{-1} f(S_1) = S_1$

therefore $x = f^{-1}(z)$ in $L_{S_1}(T_1)$

therefore f is 1-1 map of $L_{S_1}(T_1)$ to $L_{S_2}(T_2)$ which clearly preserves ring operations

Q.E.D.

Theorem 2.3.24: A ring R with unity satisfies the $\frac{\text{ACC}}{\text{DCC}}$ for left ideals iff it satisfies the $\frac{\text{ACC}}{\text{DCC}}$ for left idealisers.i.e.

Proof: If A has ACC for left ideals, then each $L_{S_1}(T)$ (where S_1 form $\frac{\text{ascending}}{\text{descending}}$ chain of ideals, and T subset of R) being a left ideal, it has ACC for the left-idealisers.

Conversely let A have ACC for the left idealizers. Then, for any ascending sequence $L_1 \subseteq L_2 \subseteq L_3 \subseteq$ - - - - of left ideals, we have an ascending sequence of left idealizers

$$L_{L_1}(1) \subseteq L_{L_2}(1) \subseteq \text{------}$$ which breaks. Hence ACC for the ideals follows. Similarly for the DCC.

Q.E.D.

If we define the $\frac{\text{ACC}}{\text{DCC}}$ for idealisers with respect to right ideals T, then we have anologously.

Theorem 2.3.25: A ring with unity has $\frac{\text{ACC}}{\text{DCC}}$ for left ideals iff it has $\frac{\text{DCC}}{\text{ACC}}$ for right idealisers.

2.4.

Let R be a commutative unitary ring and S be an ideal in R, T a subset of R. For any subsets T_i of R, define that T_1 is equivalent to T_2 iff $I_S(T_1) = I_S(T_2)$ i.e. $T_1 \sim T_2 \Leftrightarrow I_S(T_1) = I_S(T_2)$. This is an equivalence relation and we get a partition of the power set $\wp(R)$ of R into equivalence classes. We shall discuss here only those T_i's which consists of single element of R, we have then

If r_1, r_2 are in R then $r_1 \sim r_2 \Leftrightarrow I_S(r_1) = I_S(r_2)$

under this equivalence relation the ring R can be partitioned into equivalence classes with respect to any ideal S of R.

Definition 2.4.1: The class number of an ideal S of R is the number of distinct equivalence classes of R with respect to S, under the equivalence relation $r_1 \sim r_2 \Leftrightarrow I_S(r_1) = I_S(r_2)$.

The natural questions that arise in this connection are

(i) when is the class number of S finite?

(ii) under what conditions will the class number of two ideals equal?

We shall be specially interested in prime ideals and the ideals which can be written as the product of prime ideals. In particular for principal ideal domains we answer both the questions asked above. We start proving.

Theorem 2.4.2: The class number of an ideal is 2 iff it is prime.

Proof: Let S be a prime ideal of R.

Suppose r is in S then since S is an ideal zr is in S for every z belonging to R hence $I_S(r) = R$. Thus for r_1, r_2 in S we have

$I_S(r_1) = I_S(r_2) = R$ i.e. $r_1 \sim r_2$. So elements of S are all in one equivalence class.

Now if r is not in S then z is in $I_S(r)$ implies z.r is in S. Since r is not in S and S is prime we must have z in S. Conversely suppose z is in S then z.r is in S since S is an ideal and we have

$I_S(r) = S$ for every r not in S. That is all those elements of R which are not in S belong to one equivalence class. Thus we have two distinct equivalence classes and hence the class number of S in 2.

Conversely let the class number of S be 2. Then r in S implies $I_S(r) = R$ as above so all elements of S belong to one class. If r is not in S, then $I_S(r) = S^*$ and S^* can not be whole of R because 1 is in R so $1.r = r$ which is not in S.

But $I_S(1) = S$ and 1 is not in S. So 1 and r must be equivalent so $S^* = S$. This implies that for all r not in S and z.r in S we have z in S. So S is prime.

Q.E.D.

Theorem 2.4.3: Class number of an ideal S is 1 iff S = R.

Proof: If r is in S then $I_S(r) = R$. In fact for every r we have

$$I_S(r) = R = I_S(1) = S.$$

Q.E.D.

Corollary 2.4.4: Every ideal, not a prime, has class number $\geq\!\!\!\!/\;\; 2$.

*****Theorem 2.4.5:** If $S = P^n$, where P is a prime ideal of a principal ideal domain R then class number of S is n + 1.

Proof: (1) Suppose r is not in P.

Now z belongs to $I_S(r)$ implies zr belongs to $S = P^n \subseteq P$.

*: In the process of proof of this theorem [2.4.5] we have assumed tha[t] $zr \in P^n$ and $r \notin P$ implies $z \in P^n$ and similar assertions that $zr \in$ [...] $r \in P^i$ but $r \notin P^{i+1}$ for $i = 1, 2, \ldots, n-1$ implies $z \in P^{n-i}$ [...]

Since R is a principal ideal domain and r is not in P so z must be in P^n i.e. $I_S(r) \subseteq P^n$.

Suppose z is in P^n then zr is in P^n so z is in $I_S(r)$ i.e. $P^n \subseteq I_S(r)$.

Hence $I_S(r) = P^n$ for every r not in P. So all r in R which are not in P belong to one equivalence class

(ii) Suppose r is in P but not in P^2.

Now z in $I_S(r)$ implies zr in P^n, but r is not in P^2 so z must be in P^{n-1} i.e. $I_S(r) \subseteq P^{n-1}$. Now z in P^{n-1}, r in P implies zr is in P^N so $P^{n-1} \subseteq I_S(r)$ and we have $I_S(r) = P^{n-1}$. Thus the set of all elements in P but not in P^2 lie in the same equivalence class. For all such elements r in R we have $I_S(r) = P^{n-1}$.

(iii) Suppose r is in P^2 but not in P^3.

As above we can prove that $I_S(r) = P^{n-2}$.

All such r are in one equivalence class

Similarly we can prove that if r is in P^i but not in P^{i+1} then $I_S(r) = P^{n-i}$ i.e. they belong to the same equivalence class. Thus for i = 1,2,3,-----,n-1 we have n-1 equivalence classes. Now if r is in P^n the $I_S(r) = R$. So all such r are in one equivalence class. In short (n+1) equivalence classes are given as follows:

(i)	r not in P	implies	$I_S(r) = P^n$.
(ii)	r in P but not in P^2	implies	$I_S(r) = P^{n-1}$.
(iii)	r in P^2 but not in P^3	implies	$I_S(r) = P^{n-2}$.
------	------	------	------
------	------	------	------
(n-1)	r in P^{n-2} but not in P^{n-1}	implies	$I_S(r) = P^2$

(n) r in P^{n-1} but not in P^n implies $I_S(r) = P$

(n+1) r in P^n implies $I_S(r) = R$

Q.E.D.

*Theorem 2.4.6: If $S = P_1 P_2 P_3 \ldots P_t$, where P_i are distinct prime ideals in a principal ideal domain R then the class number of S is equal to 2^t.

Proof: If $t = 1$ then theorem 2.4.2 gives the result and the equivalence class of R are just P_1 and complement of P_1 in R.

If $t = 2$ then $S = P_1 P_2$.

(i) Suppose r is not in P_2 then there are two possibilities r may be in P_1, may not be in P_1.

(a) Suppose r is not in P_2 but r is in P_1.

z in $I_S(r)$ implies $z.r$ is in $S = P_1 P_2 = P_1 \cap P_2$. Since r is not in P_2 so z must be in P_2 i.e. $I_S(r) \subseteq P_2$.

Now z P_2 and r is in P_1 so $zr = rz$ is in $P_1 P_2$ and hence z is in $I_S(r)$. We have $P_2 \subseteq I_S(r)$. Thus $I_S(r) = P_2$.

(b) Suppose r is not in P_2 and r is also not in P_1.

z in $I_S(r)$ implies zr is in $S = P_1 P_2 = P_1 \cap P_2$ i.e. zr is in P_1 and P_2 both and r is in none so z is in P_1 as well as P_2 i.e. z is in $P_1 \cap P_2$. Obviously we have $P_1 P_2 \subseteq I_S(r)$ so $I_S(r) = P_1 P_2$.

(ii) Suppose r is in P_2 then again there are two possibilities r may be in P_1 or may not be.

(a) Suppose r is in P_2 and also in P_1.

Then r is in $P_1 P_2 = P_1 P_2 = S$ and hence $I_S(r) = R$.

*: Proof of $P_1 P_2 = P_1 \cap P_2$ is an easy consequence of theore

(b) Suppose r is in P_2 but not in P_1.

Then as in (a) of (1) we have $I_S(r) = P_1$.

Thus for r in P_2 there are as may equivalence classes as the class number of P_1. Also r not in P_2 gives as many equivalence classes as the class number of P_1. Thus class number of $S = P_1P_2$ is twice that of P_1 i.e. $2.2 = 2^2$.

Suppose now $S = P_1P_2P_3$

(i) Corresponding to r in P_3 there are as many equivalence classes as the class number of the ideal P_2P_3 in R

(ii) Corresponding to r not in P_3 also there are as many equivalence classes as the class number of the ideal P_2P_3 in R. So class number of $S = P_1P_2P_3$ is twice the class number of P_1P_2 i.e. $2.2^2 = 2^3$.

Explicitly corresponding to distinct 2^3 equivalence classes $\{C_i\}_{i=1}^{i=8}$ with respect to the ideal $S = P_1P_2P_3$ we have

(i)	$I_S(r) = P_1$	for r in C_1
(ii)	$I_S(r) = P_2$	for r in C_2
(iii)	$I_S(r) = P_3$	for r in C_3
(iv)	$I_S(r) = P_1 \cap P_2$	for r in C_4
(v)	$I_S(r) = P_1 \cap P_3$	for r in C_5
(vi)	$I_S(r) = P_2 \cap P_3$	for r in C_6
(vii)	$I_S(r) = P_1 \cap P_2 \cap P_3$	for r in C_7
(viii)	$I_S(r) = R$	for r in C_8

Now assume that the result is true for t-1 distinct prime factors i.e. for $S_1 = P_1P_2P_3\text{-----}P_{t-1}$, the class number of S_1 is 2^{t-1}.

Then let $S = P_1P_2P_3\text{-----}P_{t-1}P_t = S_1 . P_t$

Again we will get corresponding to r in P_t as many equivalence classes as the class number of $S_1 = P_1\text{-----}P_{t-1}$ i.e. 2^{t-1}. Also for r not in P_t the number of distinct equivalence classes is equal to the class number of S_1 i.e. 2^{t-1}. So the class number of $S = P_1P_2\text{-----}P_{t-1}$ is twice the class number of S_1. So the class number of S is $2.2^{t-1} = 2^t$.

Corollary 2.4.7: If $S = P_1^{m_1}\ P_2^{m_2}\text{-----}P_t^{m_t}$ is a prime-power factorization of an ideal S, then the class number of S is $(m_1+1).\ (m_2+1)\text{----}(m_t+1)$.

Proof: The result is true for t = 1 by theorem 2.4.5.

Assume that the result is true for t-1 distinct prime power factors i.e. $S_1 = P_1^{m_1}\ P_2^{m_2}\ \text{-----}P_{t-1}^{m_{t-1}}$ has its class number $(m_1+1)\ (m_2+1)\text{-----}(m_{t-1}+1)$.

Now let $S = P_1^{m_1}\text{-----}P_{t-1}^{m_{t-1}} = S_1 .\ P_t^{m_t}$.

(i) Corresponding to r not in P_t the number of distinct equivalence classes is equal to the class number of S_1 i.e. $(m_1+1)\ (m_2+1)\text{--}(m_{t-1}+1)$

(ii) Corresponding to r in P_t but not in P_t^2 the number of distinct equivalence classes is equal to the class number of S_1.

Similarly for r in P_t^2 but not in P_t^3 and so on.

Thus the total number of distinct equivalence classes of R with respect to $S = S_1P_t$ is (m_t+1) times the class number of S_1 i.e. class number of S is

$$(m_t+1)\left\{(m_1+1)\ (m_2+1)\text{----}(m_{t-1}+1)\right\} = (m_1+1)\ (m_2+1)\text{----}(m_{t-1}+1)\ (m_t+1)$$

as required

Q.E.D.

Then we have:

Corollary 2.4.8: Two ideals have the same class number iff they have the same number t of prime factors in the same set of indices $(m_1, m_2, ---m_t)$ occurring in some order.

Thus in a principal ideal domain we have been able to classify the set of all ideals in terms of their class-numbers, because in a principal ideal domain unique factorization of ideals into prime-power factors is always possible.

After this much about class number of an ideal, we shall give some more details about equivalence classes.

Theorem 2.4.9: The set of all equivalence classes of R with respect to an ideal form a multiplicative semi-group.

Proof: Let us consider equivalence classes of R with respect to an ideal S.

Let $r_1 \sim r_2$ and $r_1' \sim r_2'$

Therefore x in $I_S(r_1' r_2')$ implies $x\, r_1 r_2'$ is in S implies $x\, r_1$ is in $I_S(r_2') = I_S(r_1')$ implies $x\, r_1 r_1'$ is in S implies x is in $I_S(r_1 r_1')$.

Similarly

y in $I_S(r_1 r_1')$ implies $y r_1 r_1'$ is in S implies $y r_1$ is in $I_S(r_1') = I_S(r_2')$ implies $y r_1 r_2'$ is in S implies y is in $I_S(r_1 r_2')$. So we have $r_1 r_1' \sim r_1 r_2'$.

Similarly $r_2 r_1' \sim r_2 r_2'$.

$r_1' r_1 \sim r_1' r_2$

and $r_2'r_1 \sim r_2'r_2$.

$$'r_1r_1' \sim r_1r_2' \sim r_2'r_2.$$

So we can define uniquely the class product

$[r][t] = [rt]$ where $[r]$ denotes the equivalence class of r.

Q.E.D.

Lemma 2.4.10: If u is a unit in R, then r and ur belong to the same class.

Proof: x in $I_S(r)$ implies xr in S implies uxr in uS = S implies x in $I_S(ur)$ i.e. $I_S(r) \subseteq I_S(ur)$.

Suppose y in $I_S(ur)$ implies yur in S implies yr in $u^{-1}S = S$ implies y in $I_S(r)$ i.e. $I_S(ur) \subseteq I_S(r)$. So $I_S(r) = I_S(ur)$.

Q.E.D.

Theorem 2.4.12: Let f be an R-endomorphism of ${}_RR$ such that $f/_S$ is an automorphism. Then $f(r) = r'$ implies $r \sim r'$.

Proof: x in $I_S(r)$ implies xr in S, implies f(xr) is in S, since f is an automorphism aimplies $xf(r) = xr'$ is in S simplies x is in $I_S(r')$ i.e. $I_S(r) \subseteq I_S(r')$.

Conversely, y in $I_S(r')$ implies yr' in S implies yf(r) is in S implies f(yr) in S implies yr in S implies y in $I_S(r)$ i.e. $I_S(r') \subseteq I_S(r)$.

So $I_S(r') = I_S(r)$.

Q.E.D.

2.5: On k-Manifolds:

Let R be a commutative unitary ring satisfying ascending chain condition for its ideals, we shall say in this case that R is a ring with ACC. Ler $R[X] = R[x_1, x_2, \text{------}, x_n]$ be the polynomial ring in n indeterminates over R, then $R[X]$ has ACC. So every ideal in $R[X]$ is finitely generated. Let S be any set in $R[X]$ and k be a prime ideal of R. Let M^* be an n-dimensional R-module. We can look upon M^* as the set of all n-tuples with entries in R. Define $M = R_k(S) \subseteq M^*$ to be the set of all $m^* = (a_1, a_2, \text{------}, a_n)$ in M^* such that $f(X)$ in S implies $f(a_1, a_2, \text{------}, a_n)$ belongs to k. Given any $(a_1, a_2, a_3, \ldots, a_n)$ in M^* the set S of all $f(x)$ in $R[x]$ such that $f(a_1, a_2, \text{------} a_n)$ is in k is an ideal in $R[X]$, and $(a_1, a_2, \text{------}, a_n)$ will be in $R_k(S)$. Also if $S = \{0\}$, the zero polynomial, then $M^* = R_k(S)$.

Next let S be the set of $R[X]$ such that $M = R_k(S)$. Let $f_1, f_2, \text{----} f_r$ be all in S, then for all $g_1, g_2, \text{------}, g_r$ in $R[X]$, m in M, we have

$$g_1(m)f_1(m) + \text{------} + g_r(m)\, f_r(m) \text{ in } k, \text{ since } k \text{ is an ideal of } R.$$

Thus $M = R_k(S^*)$, where S^* is the ideal generated by S in $R[X]$. We shall say that M is the k-manifold of S^*. Clearly, if $S_1^* \subseteq S_2^*$, then $M_1 \supseteq M_2$ where $M_i = R_k(S_i^*)$ for $i = 1,2$.

The totallity of all f in $R[X]$ such that $f(m)$ belongs to k for all m in M is an ideal of $R[X]$ and contains all ideals S^* defining M. We shall call this the ideal belonging to M.

Let $M_i = R_k(S_i^*)$, $i = 1,2$, where $S_1^* = (f_1, f_2, \text{———}, f_r)$ and $S_2^* = (g_1, g_2, \text{———}, g_s)$. Then $M_1 \cup M_2 = R_k(S_0^*)$ such that $S_0^* = S_1^* \cap S_2^*$, since k is prime and $M_1 \cap M_2 = R_k(S^*)$ such that $S^* = \langle f_1, f_2, \text{———}, f_r, g_1, g_2, \text{———}, g_s \rangle$.

In general we have

Theorem 2.5.1: A finite intersection and an arbitrary union of k-manifolds are k-manifolds.

We shall say that a k-manifold is reducible it if is a union of two proper submanifolds otherwise irreducible. We now give a characterisation of irreducible manifolds.

Theorem 2.5.2: A k-manifold M is irreducible iff the ideal S^* belonging to M is prime.

Proof: If $M = M_1 \cup M_2$, where M_1, M_2 are properly contained in M, then let S_1^* and S_2^* be the ideals belonging to M_1 and M_2. Therefore there exists f_1 in S_1^* and f_2 in S_2^* such that $f_1(M)$ is not contained in k and $f_2(M)$ is not contained in k. But $f_1(M)\, f_2(M) \subseteq k$. Hence $f_1 f_2$ is in S^* while f_1 and f_2 are not in S^*. This can also be seen directly as follows:

f_1 is in S_1^*, f_2 is in S_2^* implies $f_1 f_2$ is in $S_1^* S_2^* \subseteq S_1^* \cap S_2^* = S^*$. So $f_1 f_2$ is in S^* but neither f_1 not f_2 is in S^* implies S^* is not prime.

Conversely

Let M be irreducible. Let S^* be the ideal belong to M. If S^* were not prime, then there exists f_1, f_2 not in S^* such that $f_1 f_2$ is in S^*.

Since $f_1(m)\ f_2(m)$ is in k for all m in M and k is prime we have either $f_1(m)$ is in k or $f_2(m)$ is in k for any m in M.

Let $M_1 = \{m \text{ in } M : f_1(m) \text{ in } k\}$

$M_2 = \{m \text{ in } M : f_2(m) \text{ in } k\}$

Clearly $M = M_1 \cup M_2$.

Also neither M_1 nor M_2 is empty otherwise f_1 or f_2 will be in S* against our assumption that f_1, f_2 are not in S*.

But then M is not irreducible, contradictng our assumption.

So S* must be prime.

Q.E.D.

CHAPTER - III
RELATIVE-PROJECTIVITY AND PROPERTY ρ IN GENERAL RINGS

3.1. INTRODUCTION:

In this chapter we shall give some properties of a ring R which we will need for the last chapter of this thesis.

Let R be a ring with unity-element 1 and P be a subring of R such that R is a free right module over P with a basis $\{X_i / i \in I\}$, for some index-set I. All subrings will be assumed to contain the unity-element 1 of the ring. Note that every element of R then has the form $\sum X_i p_i$ with each $p_i \in P$. We shall denote by Rad R, the Jacobson-Radical of the ring R.

It is obvious that given any left R-module $\mathcal{M}$, we can obtain the restriction $\mathcal{M}_P$ as a P-module by merely restricting

the operators to P. On the other hand, given any left P-module $\mathcal{N}$, we can form the induced module $\mathcal{N}^R = \oplus \sum X_i \otimes \mathcal{N}$ as an R-module, where the symbol $X_i \otimes \mathcal{N}$ stands for the tensor-product $X_i P \otimes_P \mathcal{N}$, and the direct-sum is not necessarily a module-sum even over P. If X_i centralises P, i.e. if $p . X_i = X_i\, p$ for each $p \in P$, then $X_i \otimes \mathcal{N}$ can be looked upon as a P-module. If this is the case for each i, then the above direct-sum becomes a direct-sum of P-modules.

The questions as to when the restriction of an irreducible module is completely reducible, and when the induced module of an irreducible module is completely reducible, have long been investigated. When R is the group-ring FG of a group G over a field F and P is the sub-group ring FH over F of a normal subgroup H of G, Clifford proved that the restriction of every irreducible module is completely reducible: [ ], page 343. As a sort of converse to this we investigate in the last chapter of this thesis as to when, in this situation, the module induced by an irreducible module will be completely reducible. Thus in Theorem , we show that this phenomenon ischaracterised by the fact that Rad R is, in a sense, the module over Rad P with the same basis $\{X_i\}$.

This last property is referred to as Property e in the sequel (to be defined explicitly below). It appears to be naturally and intrinsically related to the concept of relative-projectivity [2], in a strong sense. This will be more clear when we shall come to its applications to group-ring in 5th chapter.

3.2.

We recall that an R-module $\mathcal{M}$ is called P-projective if every R-exact sequence $0 \to \mathcal{N} \to \mathcal{L} \to \mathcal{M} \to 0$ of R-modules, for which the corresponding sequence of restricted modules $0 \to \mathcal{N}_P \to \mathcal{L}_P \to \mathcal{M}_P \to 0$ splits, is itself split over R. This is a property of the module On the other hand, we consider here the property of the subring P such that every R-exact sequence of the above type for which the corresponding sequence of restrictions splits, is itself split over R.

Definition 3.2.1: Let R, P and $\{X_i\}$ be as in 3.1 above. We say that $\{R, P\}$ has Property ρ with respect to the basis $\{X_i\}$ if $\sum X_i\, p_i \in$ Rad R implies that each $p_i \in$ Rad P.

Definition 3.2.2: We shall say that $\{R, P\}$ is a Projective-Pairing if every exact-sequence of R-modules, $0 \to \mathcal{N} \to \mathcal{L} \to \mathcal{M} \to 0$, for which the sequence of restrictions $0 \to \mathcal{N}_P \to \mathcal{L}_P \to \mathcal{M}_P \to 0$ splits over P, is itself split over R.

Following lemma will be useful in linking projective-pairing and property ρ.

Lemma 3.2.3: Let R be a ring with minimum condition. If $\mathcal{M}$ is an R-module such that annihilator of $\mathcal{M}$ in R contains Rad R, then $\mathcal{M}$ is completely reducible, and conversely.

Proof: For the converse, we note that Rad R is the intersection of the Kernels of all the irreducible representations of R and hence the intersection of the annihilators of the irreducible R-modules. Thus if $\mathcal{M}$ is a direct-sum of irreducible R-modules then Rad R certainly annihilates $\mathcal{M}$.

Next let annih. $\mathcal{M} \supseteq$ Rad R. Defining (r + rad R) m = r m, makes $\mathcal{M}$ into an R/Rad R - module. But R/Rad R is semi-simple with minimum-condition. Hence $\mathcal{M} = \oplus \sum \mathcal{M}_i$ where $\mathcal{M}_i$ are R/Rad R-irreducible submodules. Since Rad R $\subseteq$ annih. $\mathcal{M}$, so,

r. $\mathcal{M}_i$ = (r + Rad R) $\mathcal{M}_i \subseteq \mathcal{M}_i$, and each $\mathcal{M}_i$ is also an R-module.

If $\mathcal{N}_i$ is a proper R-submodule of $\mathcal{M}_i$, then r. $\mathcal{N}_i$ = (r + Rad R) $\mathcal{N}_i \subseteq \mathcal{N}_i$, since Rad R $\subseteq$ annih. $\mathcal{N}_i$ also. Thus $\mathcal{N}_i$ is a proper $R/_{\text{Rad R}}$- submodule of $\mathcal{M}_i$, contrary to the irreducibility of $\mathcal{M}_i$ over $R/_{\text{Rad R}}$. Hence each $\mathcal{M}_i$ is also R-irreducible, so that $\mathcal{M}$ is completely reducible over R.

Q.E.D.

In case the cardinality of I, the index-set of the basis $\{X_i\}$ of R over its subring P, is finite, we can show that Property ρ with respect to one basis implies the same with respect to any other basis.

Theorem 3.2.4: Let R be a ring with unity and P be a subring such that R is a right free P-module with finite-basis. Let $\{X_i\}$ and $\{y_i\}$ be any two P-bases of R. Then $\{R, P\}$ has Property ρ with respect to the basis $\{X_i\}$ if and only if $\{R, P\}$ has Property ρ with respect to the basis $\{y_i\}$.

Proof: We recall that if S is a ring and S_n denotes the ring of all n x n matrices over S, then Rad S_n = $(\text{Rad } S)_n$: [ ], page 11, Theorem 3.

Now let $\{X_i,\dots,X_n\}$ and $\{y_1,\dots,y_n\}$ be the two given bases for R over P, where $n < \infty$. Suppose $\sum X_i p_i \in$ Rad R implies that each $p_i \in$ Rad P.

Expressing y_i as linear combinations of the X_i's we may assume that,

$$(y_1,y_2,\dots,y_n) = (X_1,X_2,\dots,X_n) \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ - & - & - \\ - & - & - \\ - & - & - \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}$$

where each $p_{ij} \in$ P.

Similarly expressing X_i as linear-combinations of the y_i's, we may assume that,

$$(x_i,\dots,x_n) = (y_1,\dots,y_n) \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ - & - & - \\ - & - & - \\ - & - & - \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}$$

where each $q_{ij} \in$ P.

Substituting for the y_i's in the last equation from the first, and observing the linear-independence of the X_i's over P, we conclude that the matrices $[p_{ij}]$ and $[q_{ij}]$ are units in P_n.

Now suppose $\sum y_i q_i \in$ Rad R where each $q_i \in$ P.

Then $\sum_i \sum_j X_i p_{ij} q_j \in \text{Rad } R$, so that by assumption, $\sum_j p_{ij} q_j \in \text{Rad } P$, for each i. Hence the coefficients in the matrix-product,

$$\begin{bmatrix} p_{11} & \cdots & p_{1n} \\ - - - & - - - & - - - \\ - - - & - - - & - - - \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \cdot \begin{bmatrix} q_1 & 0 & 0 \\ - - - & - - - & - - - \\ - - - & - - - & - - - \\ q_n & 0 & 0 \end{bmatrix},$$

are all in Rad P.

Thus from the remark above, this product is in Rad P_n. Since the first factor is a unit in P_n, the second factor is necessarily in Rad P_n = (Rad P)$_n$. This implies that each q_i is in Rad P. Hence $\{R, P\}$ has Property ρ with respect to the basis $\{y_i\}$.

The symmetry of the above argument in the bases $\{X_i\}$ and $\{y_i\}$, then gives us the required result.

Q.E.D.

To prove a transitivity relation for Property ρ, let S, P be two subrings of R such that $S \subseteq P$, P is a right free S-module with finite basis $\{y_i\}$ and R is a right free P-module with finite basis $\{X_i\}$. In view of Theorem 3.2.4 above, we can omit the reference to basis in considering Property ρ for such pairs $\{R, P\}$, $\{P, S\}$ and $\{R, S\}$. We then have:

Theorem 3.2.5: If $\{R, P\}$ has Property ρ and $\{P, S\}$ has Property ρ, then $\{R, S\}$ has Property ρ.

Proof: It is easy to verify that if $\{X_i\}$ forms a finite p-basis of R and $\{y_i\}$ and S-basis of P then $\{X_i y_i\}$ forms a finite S-basis

of R. Now $\sum_i \sum_j X_i\, y_j\, s_{ij} \in \text{Rad } R$ with each $s_{ij} \in S$, implies that $\sum_j y_i\, s_{ij} \in \text{Rad } P$ for each i, since $\{R, P\}$ has Property ϱ with respect to the basis $\{X_i\}$. The latter, in turn, implies that each $s_{ij} \in \text{Rad } RS$ in view of Property ϱ for $\{P, S\}$ with respect to the basis $\{y_i\}$.

Thus, in view of Theorem 3.2.4, since R is a free right module over S with a finite basis $\{X_i\, y_j\}$, we obtain Property ϱ for $\{R, S\}$ with respect to any basis.

Q.E.D.

We next illustrate the connection between projective-pairing and Property ϱ. Though this will not give the complete equivalence of these two concepts, our achievement will come very near to it.

Theorem 3.2.6: Let R be a ring with unity which is a right free module over a subring P having minimum condition, with finite basis $\{X_i / i \in I\}$, $X_1 = 1$. Suppose for each $p \in P$, $p \cdot X_i = X_{p(i)} \cdot \sigma_i(p)$ where $i \to p(i)$ induces a permutation on the index-set I, and σ_i are automorphisms of the ring P.

Then Projective-pairing of $\{R, P\}$ implies Property ϱ for $\{R, P\}$.

Proof: Let $\mathfrak{M}$ be an arbitrary left P-module. Then we construct the induced R-module $\mathfrak{M}^R = \oplus \sum_{i \in I} X_i \otimes \mathfrak{M}$. Looking upon P as a set of permutations on I, let C(i) be the P-cycle to which i belongs. Thus $j \in C(i)$ implies that there is a $p \in P$ such that $p(i) = j$.

Put $W_i = \oplus \sum_{j \in C(i)} X_j \otimes \mathfrak{M}$.

It is not difficult to verify that each W_i is a left P-module and $\mathcal{M}^R = \oplus \sum W_i$ as a direct-sum of p-modules.

Now $p \in$ Rad P implies that $p \cdot W_i = p \cdot \sum_{j \in C(i)} X_j \otimes \mathcal{M}$

$= \sum_{j \in C(i)} X_{p(j)} \otimes \sigma_j(p) \mathcal{M} = 0$, since $\sigma_j(p) \in$ Rad P and $\mathcal{M}$ is P-irreducible.

Thus Rad P $\subsetneq$ annih. W_i in P. Then by Lemma 2, each W_i is completely reducible as a P-module. Hence $\mathcal{M}^R$ is completely reducible as a P-module.

Now let $0 \to \mathcal{N} \to \mathcal{M}^R \to \mathcal{L} \to 0$ be any R-exact sequence. Then this splits as a P-exact sequence since $\mathcal{M}^R$ is completely reducible over P.

As $\{R, P\}$ is a projective-pairing, so this sequence splits as an R-exact sequence also. Thus $\mathcal{M}^R$ is a completely reducible R-module.

Finally let $\sum X_i\, p_i \in$ Rad R, where each $p_i \in P$. Then from the complete-reducibility of $\mathcal{M}^R$, we have $(\sum X_i\, p_i)\, \mathcal{M}^R = 0$.

In particular, $(\sum X_i\, p_i)\,(1 \otimes m) = 0$ for every $m \in \mathcal{M}$.

Hence $\sum X_i \otimes p_i\, m = 0$ for every $m \in \mathcal{M}$.

This implies that $p_i\, \mathcal{M} = 0$ for each i.

Since $\mathcal{M}$ was an arbitrary P-irreducible module, so we conclude that each $p_i \in$ Rad P.

This proves that $\{R, P\}$ has property ρ.

Q.E.D.

Though the converse implication of this theorem is not yet completely obtainable, we almost get so in the following:-

Theorem 3.2.7: Let R be a free right module over a subring P, with finite basis $\{X_i\}$, and let R have minimum condition. If $\{R, P\}$ has Property ρ, then every R-exact sequence $0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{L} \to 0$ with $\mathcal{N}_\rho$ and $\mathcal{L}_\rho$ completely reducible over P, that splits over P, also splits over R.

Proof: If the sequence splits over P, then $\mathcal{M}_\rho \cong \mathcal{N}_\rho \oplus \mathcal{L}_\rho$ as a P-module direct-sum. Hence $\mathcal{M}_\rho$ is completely reducible over P.

Now $\sum X_i\ p_i \in$ Rad R implies $(\sum X_i\ p_i)\mathcal{M} = \sum X_i\ (p_i \mathcal{M}) = 0$, since each $p_i \in$ Rad P by Property ρ and $\mathcal{M}$ is completely reducible over P. Thus Rad R $\subseteq$ annih. $\mathcal{M}$ in R. Then by Lemma 3.2.3 above, $\mathcal{M}$ is completely reducible over R. But this implies the splitting of the given R-exact sequence.

Q.E.D.

CHAPTER - IV
AUGMENTATION TECHNIQUES IN THE STUDY OF GROUPS

4.1 INTRODUCTION:

Though the systematic exploitation of the augmentation idea seems to have been initiated in connection with Burnside-problem of groups by Magnus [1935] , Cohn [1952] , Lazard [1954] and Jennings [1955] the genus was certainly implicit in the works of Frobenius and Wedderburn. Recent works of Deskins [1956] , Connell [1963] , Coleman [1962] and Losey [1960] contain further evidence of the importance of augmentation techniques in the study of groups and group rings. In [1966] Sinha generalized the idea of relative augmentations and showed their connections with the radicals and

the representation theory. The central theme of this chapter is the study of groups and group-rings through augmentation techniques.

Now let R be an associative ring with identity, and let G be any multiplicative group. Let $A = RG$ be the group ring of G over R. Define augmentation map $\mathfrak{a} : L(G) \longrightarrow L_r(A)$ from the lattice of subgroups of the group G to the lattice of right ideals of the group ring A as follows:

For any subgroup H of G, $\mathfrak{a}(H)$ = the right ideal generated by $\{1-h : h \in H\}$ in A.

Define the inverse augmentation map $\mathfrak{a}^{-1} : L_r(A) \longrightarrow L(G)$ by $\mathfrak{a}^{-1}(J) = \{g \in G : 1-g \in J\}$.

Then $\mathfrak{a}(G) = \Delta$ is called the fundamental (or Magnus or augmentation) ideal of A. Let $\delta : A \longrightarrow R$ such that $\delta(a = \sum r_g g) = \sum r_g . \delta$ is called norm epimorphism. It is known that $\mathfrak{a}(G) = \Delta = \{a \in A : \text{if } a = \sum r_g g \text{ then } \delta(a) = 0\} = \operatorname{Ker} \delta$. Define n^{th} dimension subgroup modulo R of G to be $D_n = D_n(G \ R) = \{g \in G : g \equiv 1 \operatorname{Mod} \Delta^n\} = \mathfrak{a}^{-1}(\Delta^n)$. Then the dimension subgroups of G form a descending a central series $G = D_1 \trianglerighteq D_2 \trianglerighteq D_2 \trianglerighteq D_3 \trianglerighteq$ ———— of fully invariant subgroups of G. Moreover $(D_n, D_m) \subseteq D_{n+m}$ and so $G_n \subseteq D_n$ for every n. It is known that the dimension subgroups modulo Z, the ring of rational integers, coincide with the terms of the lower central series for free groups. It has been conjectured that the dimension subgroups modulo Z of any group G are exactly the subgroups of the lower central series of G. This is known as "Dimension conjecture". Cohn [1952], Lazard [1954] and Losey [1960] have contributed much in affirmative, but could not

solve completely.

The purpose of dimension conjecture is to calculate the factors of the lower central series of G directly from the group ring. In section 4.2 we have tried this problem from a totally different view point. Instead of studying dimension subgroups we have taken $A = A_1 \trianglerighteq A_2 \trianglerighteq A_3 \trianglerighteq$ ——, the lower central series of the group ring $A = RG$ itself and then we have considered the series $G = \bar{a}^{1}(A_1) \trianglerighteq \bar{a}^{1}(A_2) \trianglerighteq \bar{a}^{1}(A_3) \trianglerighteq \cdots$ ———which turns out to be the descending central series of G.

In particular we prove that $a(G_i) \subseteq A_i$ for every i and $a(G_2) = A_2$. We give a counter example to prove that $A_3 \neq a(G_3)$. Then we prove that $\{\bar{a}^{1}(A_i)\}$ is a decreasing series of normal subgroups of G such that

(i) $\left(\bar{a}^{1}(A_i) / \bar{a}^{1}(A_{i+1})\right)$ is abelian.

(ii) $(\bar{a}^{1}(A_i), \bar{a}^{1}(A_j)) \subseteq \bar{a}^{1}(A_{i+j})$.

Some other important results are also in this section.

In Section 4.3 we use quasi-regularity to isolate and characterize the images $\bar{a}^{1}(I)$ in G of an ideal I contained in the fundamental ideal of the integral group ring Z G having only trivial units (e.g. group ring of an ordered group). We have proved in this case that $\bar{a}^{1}(I) = \{1 + r : r$ is quasi-regular in $I\}$. We further prove that r in $\mathfrak{z}(A) \cap I$ implies r is quasi regular in I, where I is a proper ideal of A and $\mathfrak{z}(A)$ is the centre of A.

In Section 4.4 we have generalized the idea of augmentation map and studied the subgroups of the group G, with respect to different types of augmentation maps. In [7] Deskins and in [18] Sinha generalized the idea of augmentation map as follows:

$\overline{a}_l(H) = \bigcup_{H_i} a_l(H_i)$, where H_i runs through all conjugates of H in G.

$\underline{a}_l(H) = \bigcap_{H_i} a_l(H_i)$ " " " "

where $\overline{a}_l$ and $\underline{a}_l$ are upper and lower augmentation maps from the lattice of subgroups of the group G to the lattice of ideals of the group ring A = RG. We have further generalized as follows.

Let $\mathfrak{A} = \mathfrak{A}(G)$ = the group of all automorphisms of G.

$\mathfrak{I} = \mathfrak{I}(G)$ = " " inner automorphisms of G.

$\mathfrak{B} = \mathfrak{B}(G)$ = the subgroup of the group $\mathfrak{A}$ containing $\mathfrak{I}$.

Now define $\overline{a}^{l}_{\mathfrak{B}}(H) = \bigcup_{\beta \in \mathfrak{B}} a_l(H^{\beta})$ call $\overline{a}^{l}_{\mathfrak{B}}$ to be left upper augmentation of H with respect to $\mathfrak{B}$ where $a_l(H^{\beta})$ denotes the left ideal generated by the set of all elements $\{1-h^{\beta} : \beta$ in $\mathfrak{B}$, h in H$\}$. Then we prove that

"If H in L (G) is finite then $\underline{a}_{l\mathfrak{B}}(H)$ has a non-trivial two-sided annihilator in A. Conversely $\overline{a}_{l\mathfrak{B}}(H)$ has a non-trivial right or left annihilator in A, then H is finite".

Finally we have proved:

(a) If H is a finite subgroup of a p-Sylow subgroup of G and Char R=p^e, $e \geq 1$, then $\underline{a}_{l\mathfrak{B}}(H)$ is nil.

(b) Conversely if R has strict characteristic p and H is any subgroup of G, then $\overline{a}_{l\mathfrak{B}}(H)$ is nilpotent only if H is a finite subgroup of a p-Sylow subgroup of G.

Lastly in Section 4.5 we give the homology and cohomology theory for special types of augmented group-rings. Augmented group rings that we deal with are related to different types of augmentation maps discussed in previous section. Also we obtain results concerning homological dimensions of different types of augmentation images.

4.2. The Central Chains in Groups and Group Rings:

Let $A = RG$ be the group ring of the group G over an associative commutative unitary ring R. Let $\mathfrak{a} : L(G) \longrightarrow L_r(A)$ be the augmentation map such that $H \in L(G)$ implies $\mathfrak{a}(H)$ = the right ideal in A generated by the set $\{1-h : h \in H\}$. Following facts about $\mathfrak{a}$ are well known See Connel [5].

4.2.1:

(i) $1-g \in \mathfrak{a}(H)$ iff $g \in H$.

(ii) If the set $\{g_i\}$ generates the subgroup H, then the right ideal generated by $\{1-g_i\}$ is $\mathfrak{a}(H)$.

(iii) $[\mathfrak{a}(H)]^{\ell} = \{a \in A : a \cdot \mathfrak{a}(H) = 0\} \neq 0$ iff H is finite.

(iv) $H_1 \neq H_2$ implies $\mathfrak{a}(H_1) \neq \mathfrak{a}(H_2)$.

(v) $H_1 \leq H_2$ implies $\mathfrak{a}(H_1) \leq \mathfrak{a}(H_2)$.

(vi) $\mathfrak{a}(H_1 \cup H_2) = \mathfrak{a}(H_1) \cup \mathfrak{a}(H_2)$ (lattice sum)

(vii) $\mathfrak{a}(H_1 \cap H_2) \subseteq \mathfrak{a}(H_1) \cap \mathfrak{a}(H_2)$.

(viii) $\mathfrak{a}(H)$ is an ideal iff H is a normal subgroup and then

$R.\frac{G}{H} \cong \frac{RG}{\mathfrak{a}(H)}$ where $\cong$ denotes canonical ring isomorphism.

Now let $\bar{\mathfrak{a}}^{1} : L_r(A) \longrightarrow L(G)$ such that $\bar{\mathfrak{a}}^{1}(J) = \{g \in G : 1-g \in J\}$ where J is a right ideal of $A = RG$.

Following facts about $\bar{\mathfrak{a}}^{1}$ and $\mathfrak{a}$ are well known (again see Connel [5])

4.2.2:

(i) J is an ideal implies $\bar{\mathfrak{a}}^{1}(J)$ is normal.

(ii) $\bar{\mathfrak{a}}^{1}$ is onto.

(iii) $J_1 \leq J_2$ implies $\bar{\mathfrak{a}}^{1}(J_1) \leq \bar{\mathfrak{a}}^{1}(J_2)$

(iv) $\bar{a}^{1}(J_1 \cap J_2) = \bar{a}^{1}(J_1) \cap \bar{a}^{1}(J_2)$

(v) $\bar{a}^{1}(J_1 \cup J_2) \supseteq \bar{a}^{1}(J_1) \cup \bar{a}^{1}(J_2)$

(vi) $\bar{a}^{1} a(H) = H$.

(vii) $a\,\bar{a}^{1}(J) \subseteq \Delta \cap J$.

(viii) $\bar{a}^{1}(J) = \bar{a}^{1}(\Delta \cap J)$.

(ix) $A/\Delta \cong R$.

Let $G = G_1 \trianglerighteq G_2 \trianglerighteq G_3 \trianglerighteq G_4 \trianglerighteq$ ------be the lower central series of G, where $G_i=(G_{i-1},G)$ recursively and for $S, T \leq G$ we have (S,T) = the subgroup of G generated by all the commutators $\{(s,t) = (sts^{-1}t^{-1} : s \in S, t \in T\}$. Since each $G_i \trianglelefteq G$ (normal in G) so each $a(G_i)$ is two-sided ideal in A. G is said to be nilpotent of class c if its lower central series ends after c steps, c minimal.

On the other hand, we have the lower central series of the group ring A = RG i.e. $A = A_1 \trianglerighteq A_2 \trianglerighteq A_3 \trianglerighteq A_4 \trianglerighteq$ ------where the terms $A_i= [A_{i-1},A]$ recursively and for $S, T \leq A$, we define $[S, T]$ = the ideal in A generated by all additive commutators

$$\{[s,t] = st - ts : s \in S, t \in T\}$$

A is said to be of finite class k if its lower central series ends after k steps, k minimal. $A_2 = [A,A]$ is called the commutator ideal of A. It is not difficult to prove that if A is of finite class then the commutator ideal A_2 is nilpotent in the associative sense [see for proof Jennings [12]] . As said earlier, the purpose of this section is to indicate the relative properties of these two central series with respect to the augmentation map.

Theorem 4.2.3: $a(G_i) \subseteq A_i$, for every i.

Proof: We have

$$ghg^{-1}h^{-1} = (gh-hg)\, g^{-1}h^{-1} + 1$$

i.e. $(g,h)-1 = (gh-hg)g^{-1}h^{-1}$ for every $g,h \in G$.

so $(g,h)-1 \in A_2$ for every $g,h \in G$

and hence $\mathfrak{a}(G_2) \subseteq A_2$.

Now assume that $\mathfrak{a}(G_i) \subseteq A_i$ for $i = 1,2,\text{------},n$, and use induction.

Note that $G_{n+1} = \langle (g,h): g \in G_n, h \in G \rangle$ and $\mathfrak{a}(G_{n+1}) = \langle 1-g: g \in G_{n+1} \rangle$

Then $ghg^{-1}h^{-1}-1 = (gh-hg)\, g^{-1}h^{-1}$

and $gh - hg = (g-1)\, h - h\, (g-1) = [g-1,h]$

Also $g \in G_n \Rightarrow g-1 \in \mathfrak{a}(G_n) \subseteq A_n$, by the induction hypothesis.

Hence $g \in G_n$, $h \in G \Rightarrow [g-1,h] \in [A_n,A] = A_{n+1}$

Thus each generator of $\mathfrak{a}(G_{n+1})$ is in A_{n+1} and so $\mathfrak{a}(G_{n+1}) \subseteq A_{n+1}$

This completes the induction.

Q.E.D.

Corollary 4.2.4: $G_i \subseteq \mathfrak{a}^{-1}(A_i)$ for every i.

Proof: From above theorem $\mathfrak{a}(G_i) \subseteq A_i$, for every i.

So $\mathfrak{a}^{-1}\mathfrak{a}(G_i) \subseteq \mathfrak{a}^{-1}(A_i)$ but $\mathfrak{a}^{-1}\mathfrak{a}(G_i) = G_i$

and hence $G_i \subseteq \mathfrak{a}^{-1}(A_i)$ for every i.

Q.E.D.

Theorem 4.2.5: $\mathfrak{a}(G_2) = A_2$.

Proof: By Theorem 4.2.3, $\mathfrak{a}(G_2) \subseteq A_2$

On the other hand, the generator a of A_2 has the form

$$a = \sum_{i=1}^{t} r_i g_i \cdot \sum_{j=1}^{t} r_j' g_j - \sum_{j=1}^{t} r_j' g_j \cdot \sum_{i=1}^{t} r_i g_i$$

where by choosing proper number of zero coeffients, we may assume that $\{g_i\}$ is a fixed finite set of elements of G arranged in some order.

Then

$$\begin{aligned} a &= \sum_i r_i \left[\sum_j g_i r'_j g_j - \sum_j r'_j g_j g_i\right] \\ &= \sum_i r_i \left[\sum_j r'_j g_i g_j - \sum_j r'_j g_j g_i\right] \\ &= \sum_i r_i \left[\sum_j r'_j (g_i g_j - g_j g_i)\right] \\ &= \sum_i r_i \left[\sum_j r'_j \{(g_i g_j g_i^{-1} g_j^{-1} - 1) g_j g_i\}\right] \\ &= \sum_i r_i \left[\sum_j r'_j \{(g_i\, g_j) - 1\} g_j g_i\right] \end{aligned}$$

which implies $a \in \mathfrak{a}(G_2)$

and so $A_2 \subseteq \mathfrak{a}(G_2)$. Thus we have

$$A_2 = \mathfrak{a}(G_2).$$

Q.E.D.

Corollary 4.2.6: $G_2 = \mathfrak{a}^{-1}(A_2)$

Proof: Obvious

Lemma 4.2.7: If G is a nilpotent group of class 2 generated by two elements x_1, x_2 such that $(x_1, x_2)-1$ is not nilpotent then $A_3 \neq \mathfrak{a}(G_3)$.

Proof: G is nilpotent of class 2 implies $G_3 = 1$ implies $\mathfrak{a}(G_3) = 0$

So I have to prove $A_3 \neq 0$

Suppose to the contrary that $A_3 = 0$ i.e. A is of finite class then the commutator ideal A_2 is nilpotent. But $A_2 = \mathfrak{a}(G_2)$ and

$(x_1, x_2)-1 \in \mathfrak{a}(G_2)$, so $(x_1\, x_2)-1 \in A_2$ but $(x_1, x_2)-1$ is given to be not nilpotent, so A_2 can not be nilpotent. Thus $A_3 \neq 0$.

Q.E.D.

*Example 4.2.8: Take $x_1 = A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

*: This example has been taken from [19].

and $x_2 = B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$x_1^2 = A^2 = I$ (identity matrix) so $A = A^{-1}$

$x_2^2 = B^2 = I$ (" ") so $B = B^{-1}$

Take the group G to be generated by these two matrices i.e. $G = \langle A, B \rangle$

Then $(A, B) - I = ABA^{-1}B^{-1} - I = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & -0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = -2I$ which

commutes with both A and B.

So $G_3 = I$ i.e. G is nilpotent group of class 2.

Further $(A,B)-I$ is not nilpotent

so by above Lemma $A_3 \neq \alpha(G_3)$

Theorem 4.2.9: $\{\alpha^{-1}(A_i)\}$ $i = 1,2,3,\ldots\ldots$ is a descending series of normal subgroups of G such that

(a) $\left(\alpha^{-1}(A_i) / \alpha^{-1}(A_{i+1})\right)$ is abelian.

and

(b) $(\alpha^{-1}(A_i), \alpha^{-1}(A_j)) \subseteq \alpha^{-1}(A_{i+j})$.

Proof: (a) Let $g \in \alpha^{-1}(A_i)$, $h \in G$, we prove more generally that $(g,h) \in \alpha^{-1}(A_{i+1})$.

Since $g-1 \in A_i$. So $(g-1)h - h(g-1) \in A_{i+1}$ whence

$gh-hg = (ghg^{-1}h^{-1}-1)\,hg \in A_{i+1}$

Since hg is unit in A and A_{i+1} is an ideal so $ghg^{-1}h^{-1}-1 \in A_{i+1}$

implies $ghg^{-1}h^{-1} \in \bar{a}^{1}(A_{i+1})$ i.e. $(g,h) \in \bar{a}^{1}(A_{i+1})$

(b) Let $g_i \in \bar{a}^{1}(A_i)$, $g_j \in \bar{a}^{1}(A_j)$

Then let $g_i = 1 + a_i$ and $g_j = 1+a_j$ where $a_i \in A_i$ and $a_j \in A_j$

Hence $g_i g_j g_i^{-1} g_j^{-1} = 1 + (g_i g_j - g_j g_i)\, g_i^{-1}\, g_j^{-1}$

$$= 1 + \{(1+a_i)\,(1+a_j) - (1+a_j)\,(1+a_i)\}\; g_i^{-1}\, g_j^{-1}$$

$$= 1 + \;(a_i a_j - a_j a_i)\, g_i^{-1}\, g_j^{-1}$$

Now $[A_i, A_j] \subseteq A_{i+j}$ and so $(a_i a_j - a_j a_i) \in A_{i+j}$

So $(a_i a_j - a_j a_i)\, g_i^{-1}\, g_j^{-1} \in A_{i+j}$ as A_{i+j} is an ideal.

So $(g_i, g_j) = g_i g_j g_i^{-1} g_j^{-1} \equiv 1 \bmod A_{i+j}$

so $(g_i, g_j) \in \bar{a}^{1}(A_{i+j})$

therefore

$$(\bar{a}^{1}(A_i),\; \bar{a}^{1}(A_j)) \subseteq \bar{a}^{1}(A_{i+j})$$

Q.E.D.

Corollary 4.2.10: If A is of finite class k, then G is nilpotent of class atmost k.

Proof: A is of finite class k implies $A_{k+1} = (0)$.

But we know that $G_i \subseteq \bar{a}^{1}(A_i)$ for every i

In particular $G_{k+1} \subseteq \bar{a}^{1}(A_{k+1}) = (1)$

Q.E.D.

Thus $G_{k+1} = (1)$.

Theorem 4.2.11: If $g \in \bar{a}^{1}(A_n)$ then $(g, h_1, h_2, \text{------}, h_m) \in \bar{a}^{1}(A_{m+n})$

for all $h_i \in G$ and any positive integer m.

Proof: In the proof of Theorem 4.2.9 part (a), we have already proved

$(g, h_1) \in \bar{a}^{1}(A_{n+1})$

Assume $c_{i+1} = (g_1 h_1, h_2, \text{-----}, h_i) \in \tilde{\mathfrak{a}}'(A_{n+i})$ for $i = 1, 2, \text{---}, m-1$.

And use induction we have

$$(g_1 h_1, h_2, \text{-----}, h_{m-1}, h_m) = (c_m\ h_m) = c_m h_m c_m^{-1} h_m^{-1}$$

$$= 1 + (c_m h_m - h_m c_m)\ c_m^{-1}\ h_m^{-1}$$

$$= 1 + \{(c_m - 1)\ (h_m - 1) - (h_m - 1)\ (c_m - 1)\}\ c_m^{-1}\ h_m^{-1}$$

$$= 1 + [c_m - 1,\ h_m - 1]\ c_m^{-1}\ h_m^{-1}$$

But $c_m - 1 \in A_{n+m-1}$ (by induction hypothesis) $h_m - 1 \in A$

so $[c_m - 1,\ h_m - 1] \in A_{n+m}$ and hence $[c_m - 1,\ h_m - 1]\ c_m^{-1}\ h_m^{-1} \in A_{n+m}$

Thus $(g,\ h_1, h_2, \text{-----}, h_m) \equiv 1 \bmod A_{n+m}$

i.e. $(g,\ h_1, h_2, \text{-----}, h_m) \in \tilde{\mathfrak{a}}'\ (A_{n+m})$.

Q.E.D.

Theorem 4.2.12: For any $x \in \tilde{\mathfrak{a}}'(A_n)$ and $y,\ a_i \in A$,

$$[(xy),\ a_1, a_2, \text{-----}, a_m] \equiv\ x[y, a_1, a_2, \text{-----}, a_m] \quad \text{modulo} \quad A_{n+m}$$

In particular, for $y = 1$

$$[x, a_1, a_2, \text{-----}, a_m] \equiv x\ [a_1, a_2, \text{----}, a_m] \quad \text{mod } A_{n+m},$$

Proof: $[xy, a_1] = x\ [y, a_1] + [x, a_1]\ y$

$= x\ [y, a_1] + [z, a_1]\ y$ where $x = 1+z$, $z \in A_n$ as $x \in \tilde{\mathfrak{a}}'(A_n)$

Thus $[z, a_1] \in A_{n+1}$ and

$[xy, a_1] \equiv x\ [y, a_1] \bmod A_{n+1}$

Now assume

$$[xy, a_1, a_2, \text{-----}, a_{m-1}] \equiv x\ [y, a_1, a_2, \text{-----}, a_{m-1}] \quad \text{mod } A_{n+m-1};$$

Now $[xy, a_1, a_2, \text{———}, a_{m-1}, a_m] = [x\ [y, a_1, a_2, \text{———}, a_{m-1}],\ a_m]$

$$+ [r, a_m] \text{ for some } r \in A_{n+m-1}$$

Put $s = [y, a_1, a_2, \text{———}, a_{m-1}] \in A$

$$[xy, a_1, a_2, \text{———}, a_{m-1}, a_m] \equiv [xs,\ a_m] \mod A_{n+m}$$

$$\equiv x[s, a_m] \mod A_{n+m}$$

$$\equiv x[y, a_1, a_2, \text{———}, a_m] \mod A_{n+m}$$

Q.E.D.

4.3.

Now we turn to quasi-regularity of certain ideals in the integral group ring ZG. So hence on wards $A = ZG$. In any ring R, an element $r \in R$ is said to be quasi-regular if there exists another element $s \in R$ such that $r \quad s = r + s + rs = 0$. Now I shall be giving a characterization of $\bar{a}_e^{1}(I)$ in G of ideals $I \trianglelefteq a_1(G) = \Delta$.

Theorem 4.3.1: Let $A = ZG$ be the integral group ring having only trivial units {e.g. when G is an ordered group} and $I \subseteq \Delta$. Then $\{1 + r : r$ is quasi-regular in $I\}$ is exactly the subgroup $\bar{a}_e^{1}(I)$ of G.

Proof: Let $g \in \bar{a}_1(I)$ Then $g = 1+r$ for some $r \in I$. Also $g^{-1} \in \bar{a}_e^{1}(I)$

So that $g^{-1} = 1 + r^1$ for some $r^1 \in I$.

Hence $gg^{-1} = 1 = (1+r)(1+r^1) = 1+r+r^1+rr^1$

and so $r0r^1 = r + r^1 + rr^1 = 0$ so r is quasi-regular in I.

Conversely we have

If r is quasi-regular in I with quasi- inverse $r^1 \in I$. Then $1 + r + r^1 + rr^1 = (1 + r)(1 + r^1) = 1$ so that $1 + r$ is a unit in ZG. From the assumption on ZG all its units are trivial so that $1 + r = ag$ where $g \in G$ and a is a unit in Z. Since the only units in Z are $\pm$ so

$1 + r = g$ or $1 + r = -g$

If $1 + r = g$ then $r = g-1$ since $r \in I$ so $g \in \bar{a}_1(I)$.

On the other hand if $1+r = -g$ then $r = -(1+g) \in I \subseteq \Delta$

but $\Delta = \operatorname{Ker} \delta$ where $\delta : A = ZG \longrightarrow Z$ defined by $\delta(\sum z_g g) = \sum z_g$

$\delta(r = -(1+g)) = -2$ which is not zero so $r \notin \Delta$; a contradiction

so $1 + r \neq -g$.

Hence $1 + r = g$ and $g \in \alpha^{-1}(I)$ as proved above.

Q.E.D.

Finally consider the case of a finite group $G = \{g_1=1, g_2, g_3, \text{---}, g_n\}$ and its integral group ring $A = Z\,G$. Let $Z\,(A)$ be centre of A and $Z\,(G)$ be the centre of G.

Theorem 4.3.2: For any ideal $I \trianglelefteq A$, $r \in Z\,(A) \cap I$ implies r is quasi-regular in I.

Proof: Let $a = \sum r_g g$, $r_g \in Z$, $g \in G$. We wish to find $a' \in I$ such that $a \circ a' = a + a' + aa' = 0$. We assume $a \in Z\,(A) \cap I$.

Note that is a' exists then $a' = -a - aa' \in I$.

Now if a' were to exist then for $a' = \sum r'_g\, g$, $r'_g \in Z$, $g \in G$,

$$(1+a)\,(1+a') = 1 = (1+\ r_g g)\,(1+\ r'_g\, g)$$

$$= 1 + \sum r'_g\, g + \sum r'_g\, g + \sum_{g,h} r_h\ r'_{h^{-1}g}\, g.$$

Since $a \in Z\,(A)$, we we also have

$$(1+a')\,(1+a) = 1 = 1 + \sum r_g g + \sum r'_g g + \sum_{g,h} r'_h\, r_{h^{-1}g}\, g.$$

Hence, subtracting we get.

$$\sum r_h\ r'_{h^{-1}g} = \sum r'_h\, r_{h^{-1}g}$$

or $\sum(r_h\, r'_{h^{-1}g} - r'_h\, r'_{h^{-1}g}) = 0$

or $\sum r'_g\,(r_{xg^{-1}} - r_{g^{-1}x}) = 0$, for every $x \in G$

Putting $x = g_1, g_2, \text{----} g_n$ and noting that for $g_1=1$ we only get trivial equation $0 = 0$, thus we obtain $(n-1)$ linear equations in n unknown $r'_{g_1}, r'_{g_2}, \ldots, r'_{g_n}$

$$r_{g_1}\, r'_{g_1^{-1}g} + r_{g_2}\, r'_{g_2^{-1}g} + \text{------} + r_{g_n} r'_{g_n^{-1}g}$$

$$= r'_{g_1} r_{g_1^{-1}g} + \text{------} + r'_{g_n} r_{g_n^{-1}g} .$$

$g_i = g_j^{-1} g$ implies $g_j = g.g_i^{-1}$

therefore $r'_{g_1}(r_{g_1^{-1}g} - r_{g\,g_1^{-1}}) + r'_{g_2}(r_{g_2^{-1}g} - r_{gg_2^{-1}})$

$$+ \text{--------} + r'_{g_n}(r_{g_n^{-1}g} - r_{g\,g_n^{-1}}) = 0 \text{ for all } g.$$

therefore for $g = 1$ we get trivial relation

therefore for $g = g_2, g_3, \text{-----}, g_n$

we get n-1 relations in n unknowns

$r'_{g_1}, \text{-----} r'_{g_n}$ which is solvable

Q.E.D.

4.4. Generalized Augmentation Maps:

As pointed out in the introduction of this chapter, I shall be dealing with different types of augmentation maps and their relations to the lattice of subgroups of the group G and the lattices of right, left, two-sided ideals of the group ring $A = RG$. Following notations will be used in this section:

$L(G)$ = the lattice of the subgroups of the group G.

$L_N(G)$ = the lattice of the normal subgroups of the group G.

$L_r(A)$ = the lattice of the right ideals of the group ring $A = RG$.

$L_\ell(A)$ = the lattice of the left ideals of the group ring $A = RG$.

$L(A)$ = the lattice of the two-sided ideals of the group ring $A=RG$

$\mathfrak{A}(G) = \mathfrak{A}$ = the group of all automorphisms of G.

$\mathfrak{I}(G) = \mathfrak{I}$ = the group of all inner automorphisms of G.

$\mathfrak{S}(G) = \mathfrak{S}$ = subgroup of $\mathfrak{A}$ containing $\mathfrak{I}$.

We give below the definitions of all types of augmentation maps, that I shall be considering:

Definitions 4.4.1: (i) $\mathfrak{a}_\ell : L(G) \longrightarrow L_\ell(A)$ such that for $H \in L(G)$, $\mathfrak{a}_\ell(H)$ is the left ideal generated by the set $\{1-g : g \in H\}$ in A.

$\mathfrak{a}_\ell$ is called the left augmentation map.

(ii) $\mathfrak{a}_r : L(G) \longrightarrow L_r(A)$ such that for $H \in L(G)$, $\mathfrak{a}_r(H)$ is the right ideal generated by the set $\{1-g : g \in H\}$ in A. $\mathfrak{a}_r$ is called the right augmentation map.

(iii) $\quad \mathfrak{a} : L_N(G) \longrightarrow L(A)$ such that for $H \in L_N(G)$, $\mathfrak{a}(H) = \mathfrak{a}_l(H) = \mathfrak{a}_r(H)$

$\mathfrak{a}$ is called the augmentation map, note $\mathfrak{a}(G) = \Delta$ = magnusideal

(iv) $\quad \bar{\mathfrak{a}}^l_{\mathcal{B}}(H) = \bigcup_{\beta \in \mathcal{B}} \mathfrak{a}_l(H^\beta)$ and $\bar{\mathfrak{a}}^l_{\mathcal{B}} : L(G) \longrightarrow L(A)$

$\bar{\mathfrak{a}}^l_{\mathcal{B}}$ is called the left upper augmentation map with respect to $\mathcal{B}$.

(v) $\quad \underline{\mathfrak{a}}^l_{\mathcal{B}}(H) = \bigcap_{\beta \in \mathcal{B}} \mathfrak{a}_l(H^\beta)$ and $\underline{\mathfrak{a}}^l_{\mathcal{B}} : L(G) \longrightarrow L(A)$ and

$\underline{\mathfrak{a}}^l_{\mathcal{B}}$ is called the left lower augmentation map with respect to

(vi) $\quad \bar{\mathfrak{a}}^r_{\mathcal{B}}(H) = \bigcup_{\beta \in \mathcal{B}} \mathfrak{a}_r(H^\beta)$ and $\bar{\mathfrak{a}}^r_{\mathcal{B}} : L(G) \longrightarrow L_r(A)$ and

$\bar{\mathfrak{a}}^r_{\mathcal{B}}$ is called the right upper augmentation map.

(vii) $\quad \underline{\mathfrak{a}}^r_{\mathcal{B}}(H) = \bigcap_{\beta \in \mathcal{B}} \mathfrak{a}_r(H^\beta)$ and $\underline{\mathfrak{a}}^r_{\mathcal{B}} : L(G) \qquad L_r(A)$ and

$\underline{\mathfrak{a}}^r_{\mathcal{B}}$ is called the right lower augmentation map.

We shall be using above definitions and notations freely without referring them again and again through out this section.

All these seven types of augmentations coincide iff H is admissible under all automorphisms $\beta \in \mathcal{B}$. In particular it happens always if H is a characteristic subgroup of G and so there is no difference for the terms of the lower central series $G = G_1 \trianglerighteq G_2 \trianglerighteq G_3 \trianglerighteq$ ------ of G and that is why in the previous section, we did not introduce generalized augmentation map.

First of all we prove the duality theorem which will enable us to drop the suffixes "l" and "r".

Theorem 4.4.2: Duality-Theorem : For all H in L(G), we have

$$\bar{\mathfrak{a}}^l_{\mathcal{B}}(H) = \bar{\mathfrak{a}}^r_{\mathcal{B}}(H) \text{ and } \underline{\mathfrak{a}}^l_{\mathcal{B}}(H) = \underline{\mathfrak{a}}^r_{\mathcal{B}}(H).$$

Proof: To prove $\bar{\mathfrak{a}}^l_{\mathcal{B}}(H) = \bar{\mathfrak{a}}^r_{\mathcal{B}}(H)$

$H \in L(G)$, $H^* \in L(G)$, $\mathfrak{a}_l(H) \cup \mathfrak{a}_l(H^*) = \mathfrak{a}_l(H \cup H^*)$ since $\mathfrak{a}_l(H \cup H^*)$

is merely the left ideal generated by the set union of $\{1-h : h \in H\}$ and $\{1-h^* : h^* \in H^*\}$ similarly $\mathfrak{a}_r(H) \cup \mathfrak{a}_r(H^*) = \mathfrak{a}_r(H \cup H^*)$

Here for any subset S in G, $\mathfrak{a}_\ell(S)$ and $\mathfrak{a}_r(S)$ are defined in the obvious way as left and right ideals respectively generated by the set $\{1-s : s \in S\}$. So we have

$$\bar{\mathfrak{a}}^{\ell}_{\mathcal{B}}(H) = \bigcup_{\beta \in \mathcal{B}} \mathfrak{a}_\ell(H^\beta) = \mathfrak{a}_\ell\Big(\bigcup_{\beta \in \mathcal{B}} H^\beta\Big) \text{ and similarly}$$

$$\bar{\mathfrak{a}}^{r}_{\mathcal{B}}(H) = \bigcup_{\beta \in \mathcal{B}} \mathfrak{a}_r(H^\beta) = \mathfrak{a}_r\Big(\bigcup_{\beta \in \mathcal{B}} H^\beta\Big)$$

Let $a \in \bar{\mathfrak{a}}^{\ell}_{\mathcal{B}}(H)$, then a is of the form

$a = \sum_{h \in \cup_{\beta \in \mathcal{B}} H^\beta} \sum_{g \in G} r_g\, g(h-1)$ Since $h \in \bigcup_{\beta \in \mathcal{B}} H^\beta$ so $h \in H^\beta$ for some β.

This implies there exists some $h' \in H$ such that $h'^{\beta} = h$

$$\text{so } a = \sum_{h' \in H} \sum_{g \in G} \sum_{\beta \in \mathcal{B}} r_g\, g\,(h'^{\beta} - 1)$$

$$= \sum_{h' \in H} \sum_{g \in G} \sum_{\beta \in \mathcal{B}} r_g\, (g h'^{\beta} - g)$$

$$= \sum_{h' \in H} \sum_{g \in G} \sum_{\beta \in \mathcal{B}} r_g\, (g h'^{\beta} g^{-1} - 1) g$$

$$= \sum_{h' \in H} \sum_{g \in G} \sum_{\beta \in \mathcal{B}} (g h'^{\beta} g^{-1} - 1)\, r_g g$$

$$= \sum_{h'' \in \cup_{\beta \in \mathcal{B}} H^\beta} \sum_{g \in G} (h'' - 1)\, r_g g \quad [\text{Since } \beta \in \mathcal{B} \text{ and } \mathcal{B} \geq \mathcal{I}]$$

Thus $a \in \bar{\mathfrak{a}}^{r}_{\mathcal{B}}(H)$. So $\bar{\mathfrak{a}}^{r}_{\mathcal{B}}(H) \supseteq \bar{\mathfrak{a}}^{\ell}_{\mathcal{B}}(H)$

Similarly it can be shown that $\bar{\mathfrak{a}}^{\ell}_{\mathcal{B}}(H) \supseteq \bar{\mathfrak{a}}^{r}_{\mathcal{B}}(H)$

so $\bar{\mathfrak{a}}^{\ell}_{\mathcal{B}}(H) = \bar{\mathfrak{a}}^{r}_{\mathcal{B}}(H)$.

To prove $\underline{\mathfrak{a}}^{\ell}_{\mathcal{B}}(H) = \underline{\mathfrak{a}}^{r}_{\mathcal{B}}(H)$

Let S^β denote a fixed complete system of coset representatives of H^β (for $\beta \in \mathcal{B}$) in G. Let $c^{\ell}_{\beta\gamma}$ be left cosets of H^β in G, and $c^{r}_{\beta\gamma}$ be its right cosets. Define the free left R-modules M^{ℓ}_{β} and M^{r}_{β} over the collections of symbols $\{c^{\ell}_{\beta\gamma}\}$ and $\{c^{r}_{\beta\gamma}\}$ respectively. Define maps $\varphi^{\ell}_{\beta} : A \longrightarrow M^{\ell}_{\beta}$ and $\varphi^{r}_{\beta} : A \longrightarrow M^{r}_{\beta}$ respectively by the formulas:

$$\varphi^{\ell}_{\beta}\left(\sum r_g\, g\right) = \sum r_g\, gH^\beta .$$
$$\varphi^{r}_{\beta}\left(\sum r_g\, g\right) = \sum r_g\, H^\beta g$$

Where gH^β and $H^\beta g$ are left and right cosets of H^β corresponding to g in G. Then obviously φ^{ℓ}_{β} and φ^{r}_{β} are R-module epimorphisms, we show that Kern $\varphi^{\ell}_{\beta} = \mathfrak{a}_{\ell}(H^\beta)$ and Kern $\varphi^{r}_{\beta} = \mathfrak{a}_{r}(H^\beta)$.

clearly for any $h^\beta \in H^\beta$ we have

$$\varphi^{\ell}_{\beta}\{g(h^\beta - 1)\} = \varphi^{\ell}_{\beta}(gh^\beta - g) = gh^\beta H^\beta - gH^\beta = gH^\beta - gH^\beta = 0$$

so $\mathfrak{a}_{\ell}(H^\beta) \subseteq$ Kern φ^{ℓ}_{β}.

Conversely

$$\varphi^{\ell}_{\beta}\left(\sum_{g \in G} r_g\, g\right) = \varphi^{\ell}_{\beta}\left(\sum_{g \in S^\beta} \sum_{h^\beta \in H^\beta} r_{gh^\beta}\, gh^\beta \right) = 0$$

$$\sum_{h^\beta \in H^\beta} r_{gh^\beta} = 0 \text{ for all } g \in S^\beta \text{ since } M^{\ell}_{\beta} \text{ is R-free.}$$

Hence if $\sum r_g g \in$ Kern φ^{ℓ}_{β}

$$\sum_{g \in G} r_g\, g = \sum_{g \in S^\beta} \sum_{h^\beta \in H^\beta} \{ r_{gh^\beta}\, gh^\beta - r_{gh^\beta}\, g \}$$

$$= \sum_{g \in S^\beta} \sum_{h^\beta \in H^\beta} r_{gh^\beta}\, g\, \{h^\beta - 1\}$$ which is in $\mathfrak{a}_{\ell}(H^\beta)$.

Thus we have $\mathfrak{a}_{\ell}(H^{\beta}) = \text{Kern}\ \varphi^{\ell}_{\beta}$ for each $\beta \in \mathfrak{B}$.

Similarly $\mathfrak{a}_{r}(H^{\beta}) = \text{Kern}\ \varphi^{r}_{\beta}$ for each $\beta \in \mathfrak{B}$.

Now consider

$$M^{\ell} = \oplus \sum_{\beta \in \mathfrak{B}} M^{\ell}_{\beta} \qquad \text{and} \qquad M^{r} = \oplus \sum_{\beta \in \mathfrak{B}} M^{r}_{\beta}$$

Let $\varphi^{\ell} = \oplus \sum_{\beta \in \mathfrak{B}} \varphi^{\ell}_{\beta}$ and $\varphi^{r} = \oplus \sum_{\beta \in \mathfrak{B}} \varphi^{r}_{\beta}$

The mapping $\sigma : M^{\ell} \longrightarrow M^{r}$ defined by

$\sigma\left\{\sum r_g\,(gH^{\beta})\right\} = \sum r_g\,(H^{\beta} g)$ is easily seen to be an R-module isomorphism such that $\sigma \circ \varphi^{\ell} = \varphi^{r}$

The diagramm shows these relations as

$$\begin{tikzcd}
 & A = RG \arrow[dl, "\varphi^{\ell}"'] \arrow[dr, "\varphi^{r}"] & \\
M^{\ell} = \oplus \sum_{\beta} M^{\ell}_{\beta} \arrow[rr, "\sigma"] & & M^{r} = \oplus \sum_{\beta \in \mathfrak{B}} M^{r}_{\beta}
\end{tikzcd}$$

since σ is an isomorphism

so $\text{Kern}\ \varphi^{\ell} = \text{Kern}\ \varphi^{r}$

But $\text{Kern}\ \varphi^{\ell} = \bigcap_{\beta \in \mathfrak{B}} \text{Kern}\ \varphi^{\ell}_{\beta} = \underline{\mathfrak{a}}^{\ell}_{\mathfrak{B}}(H)$

And $\text{Kern}\ \varphi^{r} = \bigcap_{\beta \in \mathfrak{B}} \text{Kern}\ \varphi^{r}_{\beta} = \underline{\mathfrak{a}}^{r}_{\mathfrak{B}}(H)$

$$\underline{\mathfrak{a}}^{\ell}_{\mathfrak{B}}(H) = \underline{\mathfrak{a}}^{r}_{\mathfrak{B}}(H)$$

Q.E.D.

In view of above duality theorem we define further

$\overline{\mathfrak{a}}^{\ell}_{\mathfrak{B}}(H) = \overline{\mathfrak{a}}^{r}_{\mathfrak{B}}(H) = \overline{\mathfrak{a}}_{\mathfrak{B}}(H)$ and call $\overline{\mathfrak{a}}_{\mathfrak{B}}$ as upper augmentation

map and similarly $\underline{a}^{\ell}_{\mathcal{B}}(H) = \underline{a}^{r}_{\mathcal{B}}(H) = \underline{a}_{\mathcal{B}}(H)$ and call $\underline{a}_{\mathcal{B}}$ as lower augmentation map.

Corollary 4.4.3: $g \in \bigcap_{\beta} H^{\beta}$ iff $1-g \in \underline{a}_{\mathcal{B}}(H)$

Proof: $g \in \bigcap_{\beta\in\mathcal{B}} H^{\beta} \Rightarrow g \in H^{\beta}$ for every $\beta\in\mathcal{B} \Rightarrow 1-g \in a_{\ell}(H^{\beta})$ for every β.

$$\Rightarrow 1-g \in \bigcap_{\beta} a_{\ell}(H^{\beta})$$

$$\Rightarrow 1-g \in \underline{a}_{\mathcal{B}}(H).$$

Now suppose $g \notin \bigcap_{\beta} H^{\beta} \Rightarrow g \notin H^{\beta}$ for atleast one $\beta\in\mathcal{B}$ which implies $1-g \notin a_{\ell}(H^{\beta}) \Rightarrow 1-g \notin \bigcap_{\beta} a_{\ell}(H^{\beta}) \Rightarrow 1-g \notin \underline{a}_{\mathcal{B}}(H)$.

Q.E.D.

Now we shall give a characterization of those subgroups H of G whose order is finite i.e. finite subgroups of G.

Theorem 4.4.4: If H in L(G) is finite then $\underline{a}_{\mathcal{B}}(H)$ has a non-trivial two-sided annihilator in A. If $\overline{a}_{\mathcal{B}}(H)$ has a non-trivial right or left annihilator in A, then H is finite.

Proof: Suppose H is finite, then all H^{β} are so. Define

$$e_{\beta} = \sum_{h\in H} h^{\beta}.$$

Now $(h^{\beta}-1)\, e_{\beta} = 0$ for every $h \in H$ and hence $e_{\beta} \in \mathcal{N}^{r}_{\beta}$ = the right annihilator of $a_{\ell}(H^{\beta})$. Then $\bigcup_{\beta} \mathcal{N}_{\beta} \neq 0$ and is in the right annihilator of $\bigcap_{\beta} a_{\ell}(H^{\beta}) = \underline{a}_{\mathcal{B}}(H) = \bigcap a_{r}(H^{\beta})$.

Similarly e_{β} is also in the left annihilator. Thus $\underline{a}_{\mathcal{B}}(H)$ has non-trivial two-sided annihilator.

Conversely, let $\mathcal{N}$ be a non-trivial right annihilator of $\overline{a}_{\mathcal{B}}(H)$. So $\mathcal{N}$ is a non trivial right annihilator of each $a_{\ell}(H^{\beta})$. In particular of $a_{\ell}(H)$ which implies H is finite.

Q.E.D.

Lemma 4.4.5: If $\bar{a}_{\mathcal{B}}(H)$ is a direct summond of $A = RG$ then H is finite.

Proof: Let $A = \bar{a}_{\mathcal{B}}(H) \oplus P$ where P is a non-zero two-sided ideal of A.

so $[\bar{a}_{\mathcal{B}}(H)]^{\ell} \supseteq P$ and $[\bar{a}_{\mathcal{B}}(H)]^{r} \supseteq P$ [where for a subset S of A

$S^{\ell} = \{x \in A : xS = 0\}$

$S^{r} = \{x \in A : Sx = 0\}$]

Now $a_{\mathcal{L}}(H) \subseteq \bar{a}_{\mathcal{B}}(H)$

$\Rightarrow [\bar{a}_{\mathcal{B}}(H)]^{\ell} \subseteq [a_{\mathcal{L}}(H)]^{\ell}$

$[\bar{a}_{\mathcal{B}}(H)]^{\ell} \neq 0$ so $[a_{\mathcal{L}}(H)]^{\ell} \neq 0$

$\Rightarrow$ H is finite.

Q.E.D.

.. Sinha in [18] proved the following theorem:

4.4.6: (i) If Char $R = p^{e}$, $e \geqslant 1$. If H is a finite subgroup of a p-Sylow subgroup of the group G, then $a_{\mathcal{B}}(H)$ is nil where $\mathcal{B} = \mathcal{L}$

(ii) Conversely if R has strict characteristic p and H is any subgroup of G, then $\bar{a}_{\mathcal{L}}(H)$ is nilpotent only if H is a finite subgroup of a p-Sylow subgroup of G".

We shall prove below that even if $\mathcal{B} \neq \mathcal{L}$, the conclusions of the theorem remain valid.

Definition: A ring R with unity 1 is said to have 'strict characteristic p' if for any integer $n \neq 0$, $n.R=0 \Rightarrow p/n$ and vice versa.

Theorem 4.4.7: (a) If H is a finite subgroup of a p-Sylow subgroup of G and Char R = p^e, $e \geq 1$, then $\underline{\mathfrak{a}}_{\mathfrak{B}}(H)$ is nil.

(b) Conversely if R has strict characteristic p and H is any subgroup of G, then $\overline{\mathfrak{a}}_{\mathfrak{B}}(H)$ is nilpotent only if H is a finite subgroup of a p-Sylow subgroup of G.

Proof: (a) Since $\mathfrak{B} \geq \mathfrak{I}$

$$\underline{\mathfrak{a}}_{\mathfrak{I}}(H) = \bigcap_{i\in\mathfrak{I}} \mathfrak{a}_{\ell}(H^{i}) \supseteq \bigcap_{\beta\in\mathfrak{B}} \mathfrak{a}_{\ell}(H^{\beta}) = \underline{\mathfrak{a}}_{\mathfrak{B}}(H)$$

Since by result of . Sinha $\underline{\mathfrak{a}}_{\mathfrak{I}}(H)$ is nil

and $\underline{\mathfrak{a}}_{\mathfrak{B}}(H) \subseteq \underline{\mathfrak{a}}_{\mathfrak{I}}(H)$ so $\underline{\mathfrak{a}}_{\mathfrak{B}}(H)$ is nil

(b) From definitions it is clear that

$$\overline{\mathfrak{a}}_{\mathfrak{I}}(H) \subseteq \overline{\mathfrak{a}}_{\mathfrak{B}}(H)$$

Suppose $\overline{\mathfrak{a}}_{\mathfrak{B}}(H)$ is nilpotent. This will be so iff for some positive integer N, $(\overline{\mathfrak{a}}_{\mathfrak{B}}(H))^N = 0$ iff for every choice of N elements $a_1, a_2, \ldots\ldots, a_N \in \overline{\mathfrak{a}}_{\mathfrak{B}}(H)$, distinct or not, we have $a_1 a_2 a_3 \ldots a_N = 0$

From this it is obvious that $\overline{\mathfrak{a}}_{\mathfrak{I}}(H)$ is nilpotent.

The result now follows from Dr. Sinha's theorem

Q.E.D.

4.5. On Augmented Group-Rings:

A left augmented ring is a triple (R, M, ϵ) where R is a unitary ring, M is a left R-module, $\epsilon : R \longrightarrow M$ is an R-epimorphism. M is called the augmentation module, ϵ augmentation map, $I = \mathrm{Kern}\,\epsilon$ - the augmentation ideal. For further details one is referred to chapter VIII of [3]. We shall be limiting ourselves to the case where the base ring is always the group ring of some multiplicative group G over a unitary ring.

Let G be an arbitrary multiplicative group, R be an associative ring with unity and A=RG be the group ring of G over R. Frome previous section we have the following augmented group rings which are related to different types of augmentation maps.

4.5.1. Special types of augmented group rings:

(i) $(A = RG, M = R, \delta)$ where R is trivially an RG-module (i.e. $(\Sigma r_g g)\, r = \Sigma r_g r)$) and δ is a norm epimorphism of 4.1. In this case the augmentation ideal is Δ (the Magnus ideal) and we have the short exact sequence ,

$$0 \longrightarrow \Delta \longrightarrow A{=}RG \xrightarrow{\delta} R \longrightarrow 0$$

(ii) $(A = RG, M^{\ell}, \varphi^{\ell})$ where M^{ℓ} and φ^{ℓ} are as given in the proof of the duality theorem of the previous section. In this case, the augmentation ideal will be $\underline{a}_{\mathcal{B}}(H)$ and we have the short exact sequence,

$$0 \longrightarrow \underline{a}_{\mathcal{B}}(H) \longrightarrow A{=}RG \longrightarrow M^{\ell} \longrightarrow 0$$

Particular case is obtained by putting $\mathcal{B} = \Sigma$.

In general for every left ideal I of A we have augmented group ring.

(iii) $(A=RG, A/_I, \epsilon)$ where ϵ is the canonical epimorphism of $A \longrightarrow A/_I$.

We have the short exact sequence

$$0 \longrightarrow I \longrightarrow A=RG \longrightarrow A/_I \longrightarrow 0.$$

$(A, A/_I, \epsilon)$ we shall say the augmented group ring associated with the left ideal I. As a special case w e shall consider $I = \overline{\mathfrak{a}}_{\mathfrak{G}}(H)$.

The homology (and cohomology) theory of (1) is well known. So we shall confine ourselves to type (ii) and (iii).

We start proving certain results which will be needed in the sequel. In Lemma 4.5.2 and 4.5.3 we shall take $\mathfrak{G}=\mathfrak{S}$ and as in [18] denote $\overline{\mathfrak{a}}_{\mathfrak{G}}(H)$ by $\overline{\mathfrak{a}}_{\ell}(H)$ and denote

$\mathfrak{a}_{\mathfrak{S}}(H)$ by $\mathfrak{a}_{\ell}(H)$ and through out A=RG.

Lemma 4.5.2: For every subgroup H of G we have $A/\overline{\mathfrak{a}}_{\ell}(H) \cong R\cdot G/_{\overline{H}}$ where $\overline{H}$ is the normal hull of H in G.

Proof: From Theorem 4* of [18] we have

$$\overline{\mathfrak{a}}_{\ell}(H) = \mathfrak{a}_{\ell}(\overline{H}).$$

From (viii) of 4.2.1 we have $A/\mathfrak{a}_{\ell}(\overline{H}) \cong R.G/_{\overline{H}}$.

So we have $A/\overline{\mathfrak{a}}_{\ell}(H) = A/\mathfrak{a}_{\ell}(\overline{H}) \cong R.G/\overline{H}$

Q.E.D.

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* Theorem 4 of [18] states

"Let R be any commutative ring with unity. Then $M \trianglelefteq G$ is equal to the normal hull $\overline{H}$ of H in G iff $\overline{\mathfrak{a}}_{\ell}(H) = \mathfrak{a}_{\ell}(M)$". In the proof of this we donot need commutativity.

Corollary 4.5.3 for every subgroup H of G we have an augmented group ring $(A, \; R.G/_{\bar{H}}, \epsilon)$. with augmentation ideal $\bar{\mathfrak{a}}(H)$.

Proof: Clearly $R\cdot G/_{\bar{H}}$ is an A-module and we have from the above lemma the diagram

$$\begin{tikzcd} & A = RG \arrow[dl, "\gamma"'] \arrow[dr, "\epsilon"] & \\ A/\bar{\mathfrak{a}}(H) \arrow[rr, "\sigma"] & & R\cdot G/\bar{H} \end{tikzcd}$$

where σ is an isomorphism and $\epsilon = \sigma \circ \gamma$

Q.E.D.

In a similar way but with not so much generality we have

Lemma 4.5.4: Let R be a ring such that every finitely generated free R-module is a finite direct sum of irreducible R-modules.

Let $[G:H] < \infty$, then $A/\underline{\mathfrak{a}}(H) \cong R.\frac{G}{N}$ where $N = \cap H_i$ where H_i runs through all conjugates of H in G.

Proof: From Theorem 5 of [18] we have under these conditions

$$\mathfrak{a}(N) = \underline{\mathfrak{a}}(H).$$

and so as Lemma 4.5.2 we have

$$A/\underline{\mathfrak{a}}(H) = A/\mathfrak{a}(N) \cong R\cdot G/_N .$$

Q.E.D.

Corollary 4.5.5: Under the hypothesis in 4.5.4 we have an associated augmented group ring $(A, R\cdot G/_N, \epsilon)$ for a subgroup H of G where $N = \cap H_i$.

Now we shall give homology (cohomology) theory for the augmented group ring $(A, M^{\ell}, \varphi^{\ell})$. We have then parallel results for other augmented group ring

We have short exact sequence

$$0 \longrightarrow \underline{\mathfrak{a}}_{\mathfrak{B}}(H) \longrightarrow A \longrightarrow M^{\ell} \longrightarrow 0$$

Tensoring this sequence on the left by a right A-module Q we have an exact sequence

$$Q \otimes \underline{\mathfrak{a}}_{\mathfrak{B}}(H) \xrightarrow{\varphi} Q \otimes A \xrightarrow{\epsilon'} Q \otimes M^{\ell} \longrightarrow 0$$

$$\text{Cokern}\ \varphi = \frac{Q \otimes A}{\varphi(Q \otimes \underline{\mathfrak{a}}_{\mathfrak{B}}(H))} = \frac{Q \otimes A}{\text{Kern}\ \epsilon'} \cong Q \otimes M^{\ell}$$

So we have

4.5.6: $Q \otimes M^{\ell} \cong \text{Cokern}\ \varphi = \text{Cokern}\ (Q \otimes \underline{\mathfrak{a}}_{\mathfrak{B}}(H) \longrightarrow Q)$ as $Q \otimes A \cong Q$

Again from $0 \longrightarrow \underline{\mathfrak{a}}_{\mathfrak{B}}(H) \longrightarrow A \longrightarrow M^{\ell} \longrightarrow 0$ we have for left A-module P

$$0 \longrightarrow \text{Hom}_A(M^{\ell}, P) \longrightarrow \text{Hom}_A(A, P) \xrightarrow{\epsilon} \text{Hom}_A(\underline{\mathfrak{a}}_{\mathfrak{B}}(H), P)$$

which gives.

4.5.7: $\text{Hom}_A(M^{\ell}, P) \cong \text{Kern}\ \epsilon = \text{Kern}\ (\text{Hom}_A(A, P) \longrightarrow \text{Hom}_A(\underline{\mathfrak{a}}_{\mathfrak{B}}(H), P)$

$$= \text{Kern}\ (P \longrightarrow \text{Hom}_A(\underline{\mathfrak{a}}_{\mathfrak{B}}(H), P))$$

From the exact sequence of Homology Theorem we have

$$\longrightarrow \text{Tor}_{n+1}(Q, M^{\ell}) \xrightarrow{\partial} \text{Tor}_n(Q, \underline{\mathfrak{a}}_{\mathfrak{B}}(H)) \longrightarrow$$

$$\longrightarrow \text{Tor}_n(Q, A) \longrightarrow \text{Tor}_n(Q, M^{\ell}) \longrightarrow \text{Tor}_{n-1}(Q, \underline{\mathfrak{a}}_{\mathfrak{B}}(H))$$

$$\longrightarrow \text{Tor}_1(Q, A) \longrightarrow \text{Tor}_1(Q, M^{\ell}) \longrightarrow Q \otimes \underline{\mathfrak{a}}_{\mathfrak{B}}(H)$$

$$\longrightarrow Q \otimes A \longrightarrow Q \otimes M^{\ell} \longrightarrow 0$$

which gives

4.5.8: $\mathrm{Tor}_1(Q, M^{\ell}) \cong \mathrm{Kern}\ (Q \otimes \underline{a}_{G}(H) \longrightarrow Q \otimes A)$

$$= \mathrm{Kern}\ (Q \otimes \underline{a}_{G}(H) \longrightarrow Q)$$

Now

4.5.9: $\mathrm{Tor}_n(Q, M^{\ell}) \cong \mathrm{Tor}_{n-1}(Q, \underline{a}_{G}(H)))$ for $n > 1$.

Since $\mathrm{Tor}_{n-1}(Q,A)=0$ (as A is A-projective)

$$= \mathrm{Tor}_n\ (Q, A)$$

Now take the exact sequence of Homology Theorem for Extension functor we have

$$0 \longrightarrow \mathrm{Hom}\ (M^{\ell}, P) \longrightarrow \mathrm{Hom}\ (A, P) \longrightarrow \mathrm{Hom}\ (\underline{a}_{G}(H), P)$$
$$\longrightarrow \mathrm{Ext}^1(M^{\ell}, P) \longrightarrow \mathrm{Ext}^1(A, P) \longrightarrow \cdots\cdots$$
$$\longrightarrow \mathrm{Ext}^{n-1}(\underline{a}_{G}(H), P) \longrightarrow \mathrm{Ext}^n(M^{\ell}, P) \longrightarrow \mathrm{Ext}^n(A, P)$$
$$\longrightarrow \mathrm{Ext}^n(\underline{a}_{G}(H), P) \longrightarrow \mathrm{Ext}^{n+1}(M^{\ell}, P) \longrightarrow \mathrm{Ext}^{n+1}(A, P) \longrightarrow \cdots\cdots$$

we have

4.5.10: $\mathrm{Ext}^1(M^{\ell}, P) \cong \dfrac{\mathrm{Hom}\ (\underline{a}_{G}(H), P)}{\mathrm{Image}\ (P \longrightarrow \mathrm{Hom}(\underline{a}_{G}(H), P)}$

$$= \mathrm{Cokern}\ (P \longrightarrow \mathrm{Hom}(\underline{a}_{G}(H), P)$$

As $\mathrm{Ext}^1(A, P)=0$ since A is A projective)

4.5.11: $\mathrm{Ext}^n(M^{\ell}, P) \cong \mathrm{Ext}^{n-1}(\underline{a}_{G}(H), P)$ for $n > 1$

since $\mathrm{Ext}^n(A,P)=0$ for $n \geqslant 1$.

Similarly we can develope the further theory with the help of chapter viii of [3] . For an example the short exact sequence

$0 \longrightarrow \underline{a}_{\mathcal{B}}(H) \longrightarrow A \longrightarrow M^{\ell} \longrightarrow 0$ gives that

1 + left global dimension $\underline{a}_{\mathcal{B}}(H)$ = left global dimension M^{ℓ}

when M^{ℓ} is not A-projective.

See Page 150 [3] or Page 47 [20] .

CHAPTER - V

RELATIVE-PROJECTIVITY AND PROPERTY $\wp$ IN GROUP RINGS

5.1. INTRODUCTION:

As stated in Chapter III, we shall give here applications to grouprings of the relative-projectivity and property $\wp$. We give here a sort of converse to Clifford's Theorem [6] page 343 which says that the restriction to FH (where H is a normal subgroup of G) of wvery irreducible FG-module is completely reducible. It is well known that if G is a finite group and F is a field such that the characteristic p of F is a divisox of the order of G, then for any p-Sylow-subgroup S_p of G, the subgroup-ring FS_p enjoys the property that every FG-exact sequence of FG modules for which the corresponding sequence of restrictions to

FS_p splits then it is split over FG itself [6] i.e. $\{FG, FS_p\}$ is a projective-pairing.

The property ρ and relative-projectivity are related in a strong sense we have shown, in particular,

"Let $H \trianglelefteq G$ and R be a ring with unity-such that RG is artinian. Then $\{R, G, H\}$ has property ρ iff for every irreducible RH module $\mathfrak{M}$, the induced RG-module $\mathfrak{M}^G$ is completely reducible over RG".

We prove further

"Let R be a ring with unity and H be a subgroup of G. If $\{R, S, H\}$ has property ρ for each S in C (H) (the covering class of H in G, for definition see 5.2.2) then $\{R, G, H\}$ has property ρ Conversely if $\{R, G, H\}$ has property ρ where R is a field, then for each normal subgroup S in C(H), $\{R, S, H\}$ has property ρ ."

So many other important results are given to show the importance of property ρ in the study of group-rings.

In Section 5.4 we prove a theorem as a corollary to which follows a result of D.S. Passman [16]. Though this seems to be out of context in this chapter, yet we include it here, in order to avoid a separate chapter on it. The main theorem is

"S is a subgroup of $K_n^{\mathfrak{B}}(H)$ iff $\overline{a}_{\mathfrak{B}}(S).\Sigma_n \subseteq$ Rad FG" For the definition of $K_n^{\mathfrak{B}}(H)$ see inside 5.4.4.

5.2.

Definition 5.2.1: If G is a group and H is a subgroup of G, then let $G = \cup X_i H$ be a fixed coset decomposition of G over H. We can regard the group-ring RG as a ring with unit and RH as a subring of RG. Each element of RG is uniquely representable as $\sum X_i p_i$ where each $p_i \in RH$. Thus RG can be looked upon as a free right module over RH with basis $\{X_i\}$. We then say that $\{R, G, H\}$ has Property ρ with respect to the coset-representatives $\{X_i\}$, if $\{RG, RH\}$ has Property ρ with respect to the basis $\{X_i\}$, in the sense of Definition 1 above.

Definition 5.2.2: Let $G = \cup X_i H$ be a coset-decomposition of a group G over the subgroup H. The class $\mathcal{C}(H)$ of subgroups $H_t = \langle H, X_{i_1}, X_{i_2}, \ldots, X_{i_t} \rangle$, $t < \infty$, generated by H and a finite number t of coset-representatives $\{X_i\}$, will be called the Covering Class of H in G.

The lemma below shows that $\mathcal{C}(H)$ is well-defined by H and G.

Lemma 5.2.3: $\mathcal{C}(H)$ is independent of the choice of coset-represenations in G over H.

Proof: Let $G = \cup X_i H$ and $G = \cup y_i H$ be two coset-decomposition of G over H. Then each $y_i = X_j^{(i)} \cdot h$, for some $h \in H$, and some $X_j^{(i)}$ in the set X_i. Therefore $\langle H, y_{i_1}, \ldots, y_{i_t} \rangle = \langle H, X_j^{(i_1)}, \ldots\ldots, X_j^{(i_t)} \rangle$.

Replacing the roles of $\{y_i\}$ and $\{X_i\}$, we see atonce that $\mathcal{C}(H)$ is the same class of groups, whether defined with $\{y_i\}$ or with $\{X_i\}$.

Q.E.D.

5.3.

Let R be a ring with unity and G be a group. If H is a subgroup of finite index in G, then Theorem 3.2.4 says that if (R, G, H) has property ρ with respect to one coset-decomposition, then it has so with respect to any other coset-decomposition also. The situation in the case of group-rings is a littlemore congineal as shown in:

Theorem 5.3.1: Let H be a subgroup of G and R be a ring with unity.

If (R, G, H) has Property ρ with respect to one coset-decomposition, then it has Property ρ with respect to any other coset-decomposition.

In other wards, Property ρ for $\{R, G, H\}$ is independent of the coset-decomposition chosen, even when H is not necessarily of finite index in G.

Proof: Observe first of all that in the discrete group-ring RG, each element is a finite sum of the type $\sum r_g . g$ where $r_g \in R$, $g \in G$ and the elements of R commute with the elements of G.

Now let $\{X_i\}$, and $\{y_i\}$ be two coset-representative systems in G over H. Then each $y_i = X_i . h_i$ for some X_i, and some $h_i \in H$. Hence given $\sum y_i \, p_i$, $p_i \in RH$, we can write $\sum y_i \, p_i = \sum X_i \, h_i \, p_i$, where $h_i \, p_i \in RH$.

If $\{R, G, H\}$ has Property ρ with respect to the coset-representatives $\{X_i\}$, and $\sum y_i \, p_i \in$ Rad RG, then $\sum X_i \, h_i \, p_i \in$ Rad RG, whence each $h_i \, p_i \in$ Rad RH. Since h_i are units in RH, so this implies that each $p_i \in$ Rad RH. This gives us that $\sum y_i \, p_i \in$ Rad RG

implies each $p_i \in$ Rad RH. The symmetry of argument in $\{X_i\}$, and $\{y_i\}$ then gives us the required result.

Q.E.D.

By virtue of this theorem, through out this section we shall not mention the particular coset-representation with respect to which we have Property ρ for $\{R, G, H\}$.

Now recall that H is called a subnormal-subgroup of G [ ], if there is a chain

$$H = S_o \trianglelefteq S_1 \trianglelefteq S_2 \trianglelefteq \ldots \trianglelefteq S_n = G$$

where the symbol $S_i \trianglelefteq S_{i+1}$ stands for the statement that S_i is a normal subgroup of S_{i+1}.

If H, G and S_i are as above then we have :

Theorem 5.3.2: Let H be of finite index in G and for each i, let $\{R\,S_i, R\,S_{i-1}\}$ be a projective-pairing. Then $\{R, G, H\}$ has property ρ .

Proof: Let $\{h_i\}$ be a complete system of coset-representatives of S_1 over $H = S_o$. Then $h \in H$ implies $h. h_i = h_i. \varphi_i(h)$, where φ_i are automorphisms of H in view of the normality of H in S_1. We can extend each φ_i by linearity to an automorphism of R H. Then all the hypotheses of Theorem 3.2.6 of chapter 3 are satisfied for the ring R S_1 with the subring R S_o, so that $\{R, S_1, S_o\}$ has property ρ. Now a simple induction and the transitivity relation of Theorem 3.2.5, give us property ρ for $\{R, G, H\}$.

Corollary 5.3.3: If $H \trianglelefteq G$ and $[G:H] < \infty$ then the projective-pairing of $\{RG, RH\}$ implies property ρ for $\{R, G, H\}$.

Now we settle the question raised about the complete-reducibility of induced modules.

Theorem 5.3.4: Let $H \trianglelefteq G$ and R be a ring with unity such that RG is artinian. Then $\{R, G, H\}$ has property ρ if and only if for every irreducible RH-module $\mathcal{M}$, the induced RG-module $\mathcal{M}^G$ is completely reducible over RG.

Proof: Suppose, firstly, that for every irreducible RH-module the induced RG-module $\mathcal{M}^G$ is completely reducible over RG.

Let $G = \cup X_i H$ be a coset-decomposition of G over H, and $\sum X_i p_i \in \operatorname{Rad} RG$ where each $p_i \in RH$. The complete-reducibility of $\mathcal{M}^G$ implies that $(\sum X_i p_i)\mathcal{M}^G = 0$.

In particular for every $m \in \mathcal{M}$, an arbitrary RH-irreducible module, $(\sum X_i p_i).(1 \otimes m) = \sum X_i \otimes p_i m = 0$.

From the independence of X_i over RH, we conclude that for each i, and each $m \in \mathcal{M}$, $p_i m = 0$. Thus for every RH-irreducible module $\mathcal{M}$, $p_i \mathcal{M} = 0$ for each i. Hence each $p_i \in \operatorname{Rad} RG$, giving property ρ for $\{R, G, H\}$.

[Note that in this part of the proof we have neither made use of the normality of G in G nor of the minimum condition on RG .]

Conversely, let $\{R, G, H\}$ have property ρ. Then $\sum X_i p_i \in \operatorname{Rad} RG$, with $p_i \in RH$, implies that each $p_i \in \operatorname{Rad} RH$. Now let $\mathcal{M}$ be any irreducible RH-module. The induced RG-module $\mathcal{M}^G$ has the form $\mathcal{M}^G = \oplus \sum X_i \otimes \mathcal{M}$. Since H is normal in G, each $X_i \otimes \mathcal{M}$ is an RH-module irreducible over RH: [1] .

Also $h \in H$ implies $h X_i = X_i \cdot \varphi_i(h)$ where $\varphi_i(h) = X_i^{-1} h X_i \in H$ induces an automorphism φ_i of H for each i. We extend φ_i to RH by linearity.

Then $\sum X_i p_i \in$ Rad RG implies $(\sum X_i p_i)(X_j \otimes \mathcal{M})$

$= \sum X_i X_j \otimes \varphi_j (p_i) \mathcal{M} = \sum X_{ij} \otimes h_{ij} \varphi_j (p_i) \mathcal{M}$, where $X_i X_j = X_{ij} \cdot h_{ij}$ with $X_{ij} \in \{X_i\}$ and $h_{ij} \in H$. Since each $\varphi_j(p_i) \in$ Rad RH, and $\mathcal{M}$ is RH-irreducible so $\varphi_j(p_i)\mathcal{M} = 0$ for each i and j.

Thus $(\sum X_i p_i)\mathcal{M}^G = 0$, which shows that Rad RG $\subseteq$ annih. $\mathcal{M}^G$ in RG.

Then by lemma 3.2.3 $\mathcal{M}$ is completely reducible.

Q.E.D.

Finally we give a group-theoretic characterisation of property ρ.

Theorem 5.3.5: Let R be a ring with unity and H be a subgroup of G.

If $\{R, S, H\}$ has property ρ for each $S \in \mathcal{C}(H)$ then $\{H, G, H\}$ has property ρ.

(Conversely) If $\{R, G, H\}$ has property ρ where R is a field, then for each normal subgroup $S \in \mathcal{C}(H)$, $\{R, S, H\}$ has property ρ.

Proof: Suppose $\{R, S, H\}$ has property ρ for each $S \in \mathcal{C}(H)$.

Let $\{X_i\}$ be a complete system of coset-representatives in G over H, and $r = \sum X_i p_i \in$ Rad RG, where each $p_i \in$ RH. Observe that only a finite number of the X_i's occur in r with non-zero coefficients.

Let these be $\{X_{i_2}, \ldots\ldots, X_{i_t}\}$, $t < \infty$. Put $S = \langle H, x_{i_1}, x_{i_2}, \ldots, x_{i_t} \rangle$, $S \in \mathcal{C}(H)$.

Let y_j be a complete system of coset-representatives in G over S, where $y_1 = 1$.

Since $r \in$ Rad RG, hence there exists a quasi-inverse r^* of r in RG: [3] . This r^* satisfies the relation,

$$r^* + r - r^* . r = 0.$$

Let $r^* = \sum y_j . q_j$, where each $q_j \in$ R S.

Since $r^* + r - r^* . r = 0$, so we have on expansion,

$$y_1 (q_1 + r - q_1 . r) + \sum_{j \neq 1} y_j (q_j - q_j . r) = 0,$$

where r obviously belongs to RS.

Then from the independence of y_i over RS, it follows that each coefficient of y_j's in this last equation is independently zero.

Thus, $q_1 + r - q_1 . r = 0$; and $q_j(1 - r) = 0$ for each $j \neq 1$.

Since $1 - r$ is a unit in RG as $r \in$ Rad RG, so each $q_j = 0$ for $j \neq 1$.

Hence $r^* = q_1 \in$ RS, and $r \in$ Rad RS.

Then Property ρ for $\{R, S, H\}$ implies that each $p_i \in$ Rad RH, since $X_{i_1}, \ldots\ldots, X_{i_t}$ can be taken as part of coset-representatives in S over H.

But, then, this also gives property ρ for $\{R, G, H\}$.

Now for the other part of the theorem, let $S \in \mathcal{C}(H)$ be a normal subgroup in G, R be a field, and $\{R, G, H\}$ have property ρ. If $\mathcal{M}$ is any RG-irreducible module, then by Clifford's Theorem ; {[ ] page 343} , $\mathcal{M}_S$ is a completely reducible RS-module.

Now let $S = \cup X_i H$ be a coset-decomposition of S over G, and extend this to a coset-decomposition $G = (\cup y_i H)(\cup X_i H)$, of G over H. Let $\sum X_i p_i \in$ Rad RS where each $p_i \in$ RH. Then from the complete-reducibility of $\mathfrak{M}_S$, we conclude that $(\sum X_i p_i) \cdot \mathfrak{M} = 0$. Since this is true for an arbitrary irreducible RG-module $\mathfrak{M}$, so $\sum X_i p_i \in$ Rad RG as well.

Then Property ρ for $\{R, G, H\}$ implies that each $p_i \in$ Rad RG.

This, therefore, also implies Property ρ for $\{R, S, H\}$.

Q.E.D.

From the latter part of the proof of the above theorem we can easily extract:

Corollary 5.3.6: If R is a field and $S \trianglelefteq G$, then Rad R S $\subseteq$ Rad RG.

Corollary 5.3.7: If R is a field, $\{R, G, H\}$ has property ρ for any subgroup H in G, then $\{R, S, H\}$ has property ρ for all normal subgroups S containing H.

5.4. Character Kernels of Discrete Groups:

Let H be a subgroup of the group G, F be a field of characteristic zero, ρ be an irreducible representation of G, and Δ be the centre of the commuting ring of ρ.

Definitions:

5.4.1: ρ is said to be Finite if $\deg_\Delta \rho < \infty$.

5.4.2: ρ is said to be Strongly Finite if $\deg_F \rho < \infty$.

5.4.3: $\mathcal{B}$-Kernel of ρ in $H = \{h$ in H: $\rho(h^\beta) = 1$, for every β in $\mathcal{B}\}$.

5.4.4: $K_n^{\mathcal{B}}(H) = \bigcap \mathcal{B}$-Kern of ρ in H, where ρ varies over all irreducible representations of G such that $\deg_\Delta \rho > n$.

Lemma 5.5.5 (Schur): Δ is a field.

Lemma 5.4.6 (Amitsur): $\deg_\Delta \rho \leq n$ iff $\Sigma_n \subseteq \mathrm{Kern}\, \rho$ where Σ_n is the F-subspace of FG, Spanned by the standard Monomials $\sigma_n = \mathrm{Sgn}\,\sigma \; x_{\sigma(1)} \cdot x_{\sigma(2)} \text{--------} x_{\sigma(2n)}$, where each x_i is in G.

Now we shall prove an important theorem as a corollary of which the result of D.S. Passman [16] follows:

Theorem 5.4.7: S is a subgroup of $K_n^{\mathcal{B}}(H)$ iff $\mathfrak{a}_{\mathcal{B}}(S).\Sigma_n \subseteq \mathrm{Rad}\, FG$.

Proof: $\mathfrak{a}_{\mathcal{B}}(S) = \{\Sigma x_i(s_i^{\beta_i}-1)\, y_i : x_i, y_i \text{ in } FG,\ s_i \text{ in } S,\ \beta_i \in \mathcal{B}\}$

Since for every g in G we have $g.\Sigma_n \subseteq \Sigma_n$

So $y_i\, \Sigma_n \subseteq \Sigma_n$.

Also note that $(s_i^{\beta_i} -1)\, y_i.\Sigma_n \subseteq \mathrm{Rad}\, FG$

implies $\Sigma\, x_i\, (s_i^{\beta_i} -1)\, y_i.\Sigma_n \subseteq \mathrm{Rad}\, FG$

Therefore it suffices to prove that s in $k_n^{\mathcal{B}}(H)$ iff $(s^{\rho}-1)\Sigma_n \subseteq$ Rad FG, for every ρ in $\mathcal{B}$, and for every s in S.

For this, suppose h is in $k_n^{\mathcal{B}}(H)$ and ρ an irreducible representation of G. If $\deg_\Delta \rho > n$, then $\rho(h^{\rho}) = 1$ or $\rho(h^{\rho}-1) = 0$, for every ρ in $\mathcal{B}$ by the definition of $k_n^{\mathcal{B}}(H)$. On the other hand if $\deg_\Delta \rho \leq n$, then by Lemma 5.4.6, $\rho(\Sigma_n) = 0$. Thus in any case $\rho\{(h^{\rho}-1)\cdot\Sigma_n\} = 0$

Since ρ is arbitrary

Therefore $(h^{\rho}-1)\cdot\Sigma_n \subseteq \bigcap_{\rho \text{ irreducible}} \operatorname{Kern} \rho$ = Rad FG for every ρ in $\mathcal{B}$.

From this it follows as remarked above that

$$\overline{\mathfrak{a}}_{\mathcal{B}}(S)\cdot\Sigma_n \subseteq \text{Rad FG}.$$

Conversely,

Suppose $\overline{\mathfrak{a}}_{\mathcal{B}}(S)\cdot\Sigma_n \subseteq$ Rad FG

This implies $(s^{\rho}-1)\cdot\Sigma_n \subseteq$ Rad FG for every ρ in $\mathcal{B}$

Put $\mathfrak{I}(\Sigma_n) = \{x \text{ in FG}: x\cdot\Sigma_n \subseteq \text{Rad FG}\}$ which in the terminology of chapter II is $L_{\text{Rad FG}}(\Sigma_n)$ i.e. the Left-Idealiser of Σ_n in FG with respect to Rad FG. Then we assert that $\mathfrak{I}(\Sigma_n)$ is a two-sided ideal of FG. For, clearly $\mathfrak{I}(\Sigma_n)$ is a left ideal of FG.

Now $\left[\mathfrak{I}(\Sigma_n)\, g\right]\cdot\Sigma_n = \mathfrak{I}(\Sigma_n)\left[g\,\Sigma_n\, g^{-1}\right] g$

$= \left[\mathfrak{I}(\Sigma_n)\cdot\Sigma_n\right] g$ for every g in G since clearly $g^{-1}\Sigma_n g = \Sigma_n$

$\subseteq$ Rad FG . g

$\subseteq$ Rad FG as Rad FG is a two sided ideal of FG.

Since this is true for all g in G, hence by linearity, this is true for all x in FG. Thus $\mathfrak{I}(\Sigma_n)$ is also a right ideal of FG i.e. $\mathfrak{I}(\Sigma_n)$ is a two sided ideal in FG.

Next let ρ be an irreducible representation of G afforded by the left FG-module M and put $[\mathfrak{I}(\Sigma_n)]^{\perp} = \{m \text{ in } M: \mathfrak{I}(\Sigma_n).m = 0\}$

Since $\mathfrak{I}(\Sigma_n)$ is a two-sided ideal of FG

Therefore $[\mathfrak{I}(\Sigma_n)]^{\perp}$ is a FG-submodule of M.

Assume $\deg_{\Delta}\rho > n$.

Since M is FG-irreducible

Therefore $[\mathfrak{I}(\Sigma_n)]^{\perp} = 0$ or M.

But by Lemma 5.4.6, $\rho(\Sigma_n) \neq 0$ as $\deg_{\Delta}\rho > n$, implies $\Sigma_n \cdot M \neq 0$

so $\Sigma_n \notin$ Rad FG.

But $\mathfrak{I}(\Sigma_n)(\Sigma_n \cdot M) = [\mathfrak{I}(\Sigma_n) \cdot \Sigma_n]\, M$

$= 0$ since $\mathfrak{I}(\Sigma_n)\Sigma_n \subseteq$ Rad FG

which annahilates M irreducible

$\Sigma_n M \neq (0)$ and $\Sigma_n M \subseteq [\mathfrak{I}(\Sigma_n)]^{\perp}$ and hence $[\mathfrak{I}(\Sigma_n)]^{\perp} \neq 0$

so $[\mathfrak{I}(\Sigma_n)]^{\perp} = M$.

So $\mathfrak{I}(\Sigma_n)M = 0$

Now by hypothesis $s^{\beta} - 1$ is in $\mathfrak{I}(\Sigma_n)$ for every β in $\mathfrak{B}$, s in S.

Thefeore $(s^{\beta} - 1)\, M = 0$ and this happens for every irreducible module M

This implies $\rho(s^{\beta} - 1) = 0$ or $\rho(s^{\beta}) = 1$

Since ρ is an arbitrary with $\deg_{\Delta}\rho > n$

$\therefore$ s is in $K_n^{\mathfrak{B}}(H)$.

Q.E.D.

In the process of proof we have proved:

Corollary 5.4.8: $h \in k_n^{\mathcal{B}}(H)$ iff $(h^{\beta} - 1)\cdot\Sigma_n \subseteq \text{Rad } FG$ for every β in $\mathcal{B}$.

The result of D.S. Passman [16] follows immediately from the above theorem as

Corollary 5.4.9: If F = C the field of complex numbers and $\mathcal{B}$ = identity automorphism then

$$g \in k_n(G) \text{ iff } (1-g)\,\Sigma_n = 0$$

Proof: In this case CG is semi-simple i.e. Rad CG = 0 the rest follows from corollary 5.4.8.

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